10. DIFFERENTIATION

1. Derivatives of Some Functions

In class 11, we have studied about the derivatives of algebraic and trigonometric functions. We derived the following results.

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$	**(ii)** $\frac{d}{dx}(\sin x) = \cos x$
(iii) $\frac{d}{dx}(\cos x) = -\sin x$	**(iv)** $\frac{d}{dx}(\tan x) = \sec^2 x$
(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	**(vi)** $\frac{d}{dx}(\sec x) = \sec x \tan x$
(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	

In algebra of derivatives, we have established the following rules.

Some Rules of Differentiation

(i) $\frac{d}{dx}(u+v) = \left(\frac{du}{dx} + \frac{dv}{dx}\right)$

(ii) $\frac{d}{dx}(u-v) = \left(\frac{du}{dx} - \frac{dv}{dx}\right)$

(iii) $\frac{d}{dx}(uv) = \left(u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right)$

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}\right)}{v^2}$

DERIVATIVES OF COMPOSITE FUNCTIONS (CHAIN RULE)

(i) Let $y = f(t)$ and $t = g(x)$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$.

(ii) Let $y = f(t)$, $t = g(u)$ and $u = h(x)$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}\right)$.

This rule may be extended further on more variables.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) $\sin x^3$ *(ii)* $\cos^3 x$ *(iii)* $\tan \sqrt{x}$

SOLUTION (i) Let $y = \sin x^3$.

Putting $x^3 = t$, we get

$$y = \sin t \text{ and } t = x^3$$

$$\Rightarrow \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= (\cos t \cdot 3x^2) = 3x^2 \cos t = 3x^2 \cos x^3.$$

Hence, $\frac{d}{dx}(\sin x^3) = 3x^2 \cos x^3$.

(ii) Let $y = \cos^3 x = (\cos x)^3$.

Putting $\cos x = t$, we get

$$y = t^3 \text{ and } t = \cos x$$

$$\Rightarrow \frac{dy}{dt} = 3t^2 \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= (-3t^2 \sin x) = (-3 \sin x)t^2 = (-3 \sin x \cos^2 x).$$

Hence, $\frac{d}{dx}(\cos^3 x) = -3 \sin x \cos^2 x$.

(iii) Let $y = \tan \sqrt{x}$.

Putting $\sqrt{x} = t$, we get

$$y = \tan t \text{ and } t = \sqrt{x}$$

$$\Rightarrow \frac{dy}{dt} = \sec^2 t \text{ and } \frac{dt}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\sec^2 t \cdot \frac{1}{2\sqrt{x}}\right) = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}} \quad [\because \ t = \sqrt{x}].$$

Hence, $\frac{d}{dx}(\tan \sqrt{x}) = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}$.

EXAMPLE 2 *Differentiate each of the following w.r.t. x:*

(i) $(ax + b)^m$ *(ii)* $(2x + 3)^5$ *(iii)* $\sqrt{ax^2 + 2bx + c}$

SOLUTION (i) Let $y = (ax + b)^m$.

Putting $(ax + b) = t$, we get

$$y = t^m \text{ and } t = (ax + b)$$

$$\Rightarrow \frac{dy}{dt} = mt^{m-1} \text{ and } \frac{dt}{dx} = a$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (mt^{m-1} \times a) = mat^{m-1} = ma(ax + b)^{m-1}.$$

$$\therefore \frac{d}{dx}(ax + b)^m = ma(ax + b)^{m-1}.$$

(ii) Let $y = (2x + 3)^5$.

Putting $(2x + 3) = t$, we get

$$y = t^5 \text{ and } t = 2x + 3$$

$$\Rightarrow \frac{dy}{dt} = 5t^4 \text{ and } \frac{dt}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = 10t^4 = 10(2x + 3)^4.$$

(iii) Let $y = \sqrt{ax^2 + 2bx + c}$.

Putting $(ax^2 + 2bx + c) = t$, we get

$$y = \sqrt{t} \text{ and } t = (ax^2 + 2bx + c)$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = (2ax + 2b) = 2(ax + b)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}} \times 2(ax + b)$$

$$= \frac{(ax + b)}{\sqrt{t}} = \frac{(ax + b)}{\sqrt{ax^2 + 2bx + c}}.$$

EXAMPLE 3 *Differentiate* $\sin 3x \cos 5x$ *w.r.t. x.*

SOLUTION Let $y = \sin 3x \cos 5x = \frac{1}{2}\{2\cos 5x \sin 3x\}$

$$= \frac{1}{2}\{\sin (5x + 3x) - \sin (5x - 3x)\}$$

$$= \frac{1}{2} \cdot (\sin 8x - \sin 2x) = \frac{1}{2}\sin 8x - \frac{1}{2}\sin 2x.$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}(\sin 8x) - \frac{1}{2} \cdot \frac{d}{dx}(\sin 2x)$$

$$= \left(\frac{1}{2} \cdot 8\cos 8x - \frac{1}{2} \times 2\cos 2x\right) = (4\cos 8x - \cos 2x).$$

EXAMPLE 4 *Differentiate* $\sin 2x \sin 4x$ *w.r.t. x.*

SOLUTION Let $y = \sin 2x \sin 4x = \frac{1}{2}(2\sin 4x \sin 2x)$

$$= \frac{1}{2}\{\cos(4x - 2x) - \cos(4x + 2x)\}$$

$$= \frac{1}{2}[\cos 2x - \cos 6x].$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}(\cos 2x) - \frac{1}{2} \cdot \frac{d}{dx}(\cos 6x)$$

$$= \frac{1}{2} \cdot (-2\sin 2x) - \frac{1}{2} \cdot (-6 \sin 6x)$$

$$= (3 \sin 6x - \sin 2x).$$

EXAMPLE 5 *Differentiate* $\sqrt{\frac{1-\tan x}{1+\tan x}}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$.

Putting $\frac{(1-\tan x)}{(1+\tan x)} = t$, we get

$$y = \sqrt{t} \text{ and } t = \frac{(1-\tan x)}{(1+\tan x)}.$$

$$\therefore \quad \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}.$$

$$\text{And, } \frac{dt}{dx} = \frac{(1+\tan x) \cdot \frac{d}{dx}(1-\tan x) - (1-\tan x) \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2} = \frac{-2\sec^2 x}{(1+\tan x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = \left\{\frac{dy}{dt} \times \frac{dt}{dx}\right\} = \frac{1}{2\sqrt{t}} \times \frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^2} \times \frac{\sqrt{1+\tan x}}{\sqrt{1-\tan x}}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^{3/2}(1-\tan x)^{1/2}}.$$

EXAMPLE 6 *If* $y = \frac{1}{\sqrt{a^2 - x^2}}$, *find* $\frac{dy}{dx}$.

SOLUTION Putting $(a^2 - x^2) = t$, we get

$$y = \frac{1}{\sqrt{t}} = t^{-1/2} \text{ and } t = (a^2 - x^2)$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}} \text{ and } \frac{dt}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \left\{\frac{dy}{dt} \times \frac{dt}{dx}\right\} = \frac{-1}{2t^{3/2}} \times (-2x) = \frac{x}{t^{3/2}} = \frac{x}{(a^2 - x^2)^{3/2}}.$$

EXAMPLE 7 *If* $y = \cos^2 x^2$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $y = (\cos x^2)^2$.

Putting $x^2 = t$ and $\cos t = u$, we get

$$y = u^2,\ u = \cos t \text{ and } t = x^2$$

$$\Rightarrow \frac{dy}{du} = 2u,\ \frac{du}{dt} = -\sin t \text{ and } \frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= (2u) \times (-\sin t) \times 2x$$

$$= -4xu \sin t = -4x \cos t \sin t = -4x \cos x^2 \sin x^2.$$

$$\therefore \frac{dy}{dx} = -4x \cos x^2 \sin x^2.$$

EXAMPLE 8 *If* $y = \sin(\cos x^2)$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $x^2 = t$ and $\cos x^2 = \cos t = u$, we get

$$y = \sin u,\ u = \cos t \text{ and } t = x^2$$

$$\Rightarrow \frac{dy}{du} = \cos u,\ \frac{du}{dt} = -\sin t \text{ and } \frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \{\cos u \times (-\sin t) \times 2x\} = -2x \sin t \cos u$$

$$= -2x \sin t \cos(\cos t) = -2x \sin x^2 \cos(\cos x^2).$$

$$\therefore \frac{dy}{dx} = -2x \sin x^2 \cos(\cos x^2).$$

EXAMPLE 9 *If* $y = \sin(\sqrt{\sin x + \cos x})$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{t} = u$, we get

$$y = \sin u,\ u = \sqrt{t} \text{ and } t = (\sin x + \cos x)$$

$$\Rightarrow \frac{dy}{du} = \cos u,\ \frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = (\cos x - \sin x)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left\{\cos u \times \frac{1}{2\sqrt{t}} \times (\cos x - \sin x)\right\} = \frac{\cos \sqrt{t}}{2\sqrt{t}} \cdot (\cos x - \sin x)$$

$$= \frac{\cos(\sqrt{\sin x + \cos x})}{2\sqrt{\sin x + \cos x}} \cdot (\cos x - \sin x).$$

EXAMPLE 10 *If* $y = \sin[\sqrt{\sin\sqrt{x}}]$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $\sqrt{x} = t$, $\sin\sqrt{x} = \sin t = u$ and $\sqrt{\sin\sqrt{x}} = \sqrt{u} = v$, we get

$$y = \sin v,\ v = \sqrt{u},\ u = \sin t \text{ and } t = \sqrt{x}.$$

$$\therefore \quad \frac{dy}{dv} = \cos v;\ \frac{dv}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}};\ \frac{du}{dt} = \cos t \text{ and } \frac{dt}{dx} = \frac{1}{2\sqrt{x}}.$$

$$\text{So,} \quad \frac{dy}{dx} = \left(\frac{dy}{dv}\times\frac{dv}{du}\times\frac{du}{dt}\times\frac{dt}{dx}\right) = \left[\cos v\cdot\frac{1}{2\sqrt{u}}\cdot\cos t\cdot\frac{1}{2\sqrt{x}}\right]$$

$$= \left[\cos\sqrt{u}\cdot\frac{1}{2\sqrt{u}}\cos t\cdot\frac{1}{2\sqrt{x}}\right] \qquad [\because\ v = \sqrt{u}]$$

$$= \frac{1}{4}\cos(\sqrt{\sin t})\cdot\frac{1}{\sqrt{\sin t}}\cdot\cos\sqrt{x}\cdot\frac{1}{\sqrt{x}} \qquad [\because\ u = \sin t]$$

$$= \frac{1}{4}\cos(\sqrt{\sin\sqrt{x}})\cdot\frac{1}{\sqrt{\sin\sqrt{x}}}\cdot\cos\sqrt{x}\cdot\frac{1}{\sqrt{x}} \qquad [\because\ t = \sqrt{x}]$$

$$= \frac{\cos(\sqrt{\sin\sqrt{x}})}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}\cdot\cos\sqrt{x}.$$

EXAMPLE 11 *If* $y = \dfrac{5x}{\sqrt[3]{1-x^2}} + \sin^2(2x+3)$, *find* $\dfrac{dy}{dx}$. **[CBSE 2000C]**

SOLUTION We have

$$y = 5x(1-x^2)^{-1/3} + \sin^2(2x+3)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\{5x(1-x^2)^{-1/3}\} + \frac{d}{dx}\{\sin^2(2x+3)\}$$

$$= \left\{5x\cdot\left(\frac{-1}{3}\right)(1-x^2)^{-4/3}\cdot(-2x) + (1-x^2)^{-1/3} - 5\right\}$$

$$+ \{2\sin(2x+3)\cos(2x+3)\cdot 2\}$$

$$= \frac{10x^2}{3(1-x^2)^{4/3}} + \frac{5}{(1-x^2)^{1/3}} + 2\sin(4x+6)$$

$$= \frac{10x^2 + 15(1-x^2)}{3(1-x^2)^{4/3}} + 2\sin(4x+6)$$

$$= \frac{(15-5x^2)}{3(1-x^2)^{4/3}} + 2\sin(4x+6).$$

EXERCISE 10A

Differentiate each of the following w.r.t. x:

1. $\sin 4x$
2. $\cos 5x$
3. $\tan 3x$
4. $\cos x^3$
5. $\cot^2 x$
6. $\tan^3 x$

7. $\cot \sqrt{x}$
8. $\sqrt{\tan\ x}$
9. $(5+7x)^6$
10. $(3-4x)^5$
11. $(2x^2-3x+4)^5$
12. $(ax^2+bx+c)^6$
13. $\dfrac{1}{(x^2-3x+5)^3}$
14. $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$
15. $\sqrt{\dfrac{1+\sin\ x}{1-\sin\ x}}$
16. $\cos^2 x^3$
17. $\sec^3(x^2+1)$
18. $\sqrt{\cos\ 3x}$
19. $\sqrt[3]{\sin\ 2x}$
20. $\sqrt{1+\cot\ x}$
21. $\operatorname{cosec}^3 \dfrac{1}{x^2}$
22. $\sqrt{\sin\ x^3}$
23. $\sqrt{x \sin\ x}$
24. $\sqrt{\cot \sqrt{x}}$
25. $\cot^3 x^2$
26. $\cos(\sin \sqrt{ax+b})$
27. $\sqrt{\operatorname{cosec}(x^3+1)}$
28. $\sin 5x \cos\ 3x$
29. $\sin\ 2x \sin\ x$
30. $\cos 4x \cos 2x$

Find $\dfrac{dy}{dx}$, when:

31. $y = \sin\left(\dfrac{1+x^2}{1-x^2}\right)$
32. $y = \dfrac{(\sin\ x + x^2)}{\cot\ 2x}$
33. If $y = \dfrac{(\cos x - \sin\ x)}{(\cos x + \sin\ x)}$, prove that $\dfrac{dy}{dx} + y^2 + 1 = 0$.
34. If $y = \dfrac{(\cos x + \sin\ x)}{(\cos x - \sin\ x)}$, prove that $\dfrac{dy}{dx} = \sec^2\left(x + \dfrac{\pi}{4}\right)$.

ANSWERS (EXERCISE 10A)

1. $4\cos 4x$
2. $-5 \sin 5x$
3. $3\sec^2 3x$
4. $-3x^2 \sin\ x^3$
5. $-2\cot\ x\ \operatorname{cosec}^2 x$
6. $3\tan^2 x\ \sec^2 x$
7. $\dfrac{-1}{2\sqrt{x}} \operatorname{cosec}^2 \sqrt{x}$
8. $\dfrac{\sec^2 x}{2\sqrt{\tan\ x}}$
9. $42(5+7x)^5$
10. $-20(3-4x)^4$
11. $5(2x^2-3x+4)^4(4x-3)$
12. $6(ax^2+bx+c)(2ax+b)$
13. $\dfrac{-3(2x-3)}{(x^2-3x+5)^4}$
14. $\dfrac{-2a^2x}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$
15. $\sec x(\sec x + \tan\ x)$
16. $-3x^2 \sin\ (2x^3)$
17. $6x\ \sec^3(x^2+1) \tan\ (x^2+1)$
18. $\dfrac{-3}{2} \cdot \dfrac{\sin\ 3x}{\sqrt{\cos\ 3x}}$
19. $\dfrac{2}{3} \cdot \dfrac{\cos 2x}{(\sin\ 2x)^{2/3}}$
20. $-\dfrac{1}{2} \cdot \dfrac{\operatorname{cosec}^2 x}{\sqrt{1+\cot\ x}}$
21. $\dfrac{6}{x^3} \operatorname{cosec}^3 \dfrac{1}{x^2} \cot \dfrac{1}{x^2}$
22. $\dfrac{3}{2}x^2 \cdot \dfrac{\cos\ x^3}{\sqrt{\sin\ x^3}}$
23. $\dfrac{(x\cos x + \sin\ x)}{2\sqrt{x \sin\ x}}$

24. $\dfrac{-\text{cosec}^2\sqrt{x}}{4\sqrt{x}\sqrt{\cot\sqrt{x}}}$ **25.** $-6x\cot^2 x^2\,\text{cosec}^2 x^2$

26. $\dfrac{-a\cos\sqrt{ax+b}}{2\sqrt{ax+b}}\cdot\sin\{\sin\sqrt{ax+b}\}$ **27.** $-\dfrac{3}{2}x^2\sqrt{\text{cosec}\,(x^3+1)}\cdot\cot(x^3+1)$

28. $(4\cos 8x+\cos 2x)$ **29.** $\left(\dfrac{3}{2}\sin 3x-\dfrac{1}{2}\sin x\right)$

30. $-(3\sin 6x+\sin 2x)$ **31.** $\dfrac{4x}{(1-x^2)^2}\cdot\cos\left(\dfrac{1+x^2}{1-x^2}\right)$

32. $2(\sin x+x^2)\sec^2 2x+(\cos x+2x)\tan 2x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10A)

15. $y=\dfrac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}}\times\dfrac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}=\dfrac{(1+\sin x)}{\cos x}=(\sec x+\tan x)$

$\Rightarrow \dfrac{dy}{dx}=(\sec x\tan x+\sec^2 x)=\sec x(\sec x+\tan x).$

32. $y=(\sin x+x^2)\tan 2x$

$\Rightarrow \dfrac{dy}{dx}=(\sin x+x^2)\cdot\dfrac{d}{dx}(\tan 2x)+\tan 2x\cdot\dfrac{d}{dx}(\sin x+x^2).$

33. $y=\left(\dfrac{1-\tan x}{1+\tan x}\right)$ [on dividing num. and denom. by $\cos x$]

$\Rightarrow y=\tan\left(\dfrac{\pi}{4}-x\right)$

$\Rightarrow \dfrac{dy}{dx}=-\sec^2\left(\dfrac{\pi}{4}-x\right)$

$\Rightarrow \dfrac{dy}{dx}+y^2+1=-\sec^2\left(\dfrac{\pi}{4}-x\right)+\tan^2\left(\dfrac{\pi}{4}-x\right)+1=0.$

34. $y=\left(\dfrac{1+\tan x}{1-\tan x}\right)$

$\Rightarrow y=\tan\left(\dfrac{\pi}{4}+x\right)$

$\Rightarrow \dfrac{dy}{dx}=\sec^2\left(\dfrac{\pi}{4}+x\right).$

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2. Derivatives of Exponential and Logarithmic Functions

EXPONENTIAL FUNCTION

Let a be a real number such that $a>1$.

Then, $f(x)=a^x$ is called an ***exponential function***.

Its domain $=R$ and range $=R^+$.

When $a = e$, we have the exponential function, $f(x) = e^x$.

This is called ***natural exponential function***.

LOGARITHMIC FUNCTION

Let a be a real number such that $a > 1$.

If $a^x = b$, *we define,* $\log_a x = b$.

We say that log of x to the base a is equal to b.

$\log_{10} x$ is called ***common logarithm of x.***

$\log_e x$ is called ***natural logarithm of x*** and we denote it simply by $\log\ x$.

It is easy to verify the following results:

(i) $\log\ (xy) = (\log\ x) + (\log\ y)$ (ii) $\log\left(\frac{x}{y}\right) = (\log\ x) - (\log\ y)$

(iii) $\log\ (x^n) = n \log\ x$ (iv) $\log_a x = \frac{1}{\log_x a}$

(v) $\log_x x = 1$ and $\log 1 = 0$

Derivatives of Exponential and Logarithmic Functions

We have (i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(\log\ x) = \frac{1}{x}$.

Derivative of a^x

Let $y = a^x$. Then, $\log\ y = x \log\ a$. ...

(i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log\ a \Rightarrow \frac{dy}{dx} = y(\log\ a)$$

$$\Rightarrow \frac{dy}{dx} = a^x(\log\ a).$$

Hence, $\frac{d}{dx}(a^x) = a^x(\log\ a)$.

SUMMARY

(i) $\frac{d}{dx}(e^x) = e^x$ (ii) $\frac{d}{dx}(a^x) = a^x(\log\ a)$

(iii) $\frac{d}{dx}(\log\ x) = \frac{1}{x}$ (iv) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log\ a}$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) e^{x^2} *(ii)* e^{-3x} *(iii)* $e^{\cos x}$

SOLUTION (i) Let $y = e^{x^2}$.

Putting $x^2 = t$, we get

$$y = e^t \text{ and } t = x^2$$

$$\Rightarrow \frac{dy}{dt} = e^t \text{ and } \frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (e^t \times 2x) = (e^{x^2} \times 2x) = 2x\, e^{x^2}.$$

Hence, $\frac{d}{dx}(e^{x^2}) = 2x\, e^{x^2}$.

(ii) Let $y = e^{-3x}$.

Putting $-3x = t$, we get

$$y = e^t \text{ and } t = -3x$$

$$\Rightarrow \frac{dy}{dt} = e^t \text{ and } \frac{dt}{dx} = -3$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = -3e^t = -3e^{-3x}.$$

Hence, $\frac{d}{dx}(e^{-3x}) = -3e^{-3x}$.

(iii) Let $y = e^{\cos x}$.

Putting $\cos x = t$, we get

$$y = e^t \text{ and } t = \cos x$$

$$\Rightarrow \frac{dy}{dt} = e^t \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = (-e^t \sin x) = (-e^{\cos x} \sin x).$$

Hence, $\frac{d}{dx}(e^{\cos x}) = -e^{\cos x} \sin x$.

EXAMPLE 2 *Differentiate each of the following w.r.t. x:*

(i) $\sin(\log x)$, $x > 0$ *(ii)* $\log(\log x)$, $x > 1$

SOLUTION (i) Let $y = \sin(\log x)$.

Putting $\log x = t$, we get

$$y = \sin t \text{ and } t = \log x$$

$$\Rightarrow \frac{dy}{dt} = \cos t \text{ and } \frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{dy}{dt}\times\frac{dt}{dx}\right)=\left(\cos t\times\frac{1}{x}\right)=\cos(\log x)\times\frac{1}{x}=\frac{\cos(\log x)}{x}.$$

Hence, $\frac{d}{dx}\{\sin(\log x)\}=\frac{\cos(\log x)}{x}$.

(ii) Let $y=\log(\log x)$.

Putting $\log x=t$, we get

$$y=\log t \text{ and } t=\log x$$

$$\Rightarrow \frac{dy}{dt}=\frac{1}{t} \text{ and } \frac{dt}{dx}=\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{dy}{dt}\times\frac{dt}{dx}\right)=\left(\frac{1}{t}\times\frac{1}{x}\right)=\left(\frac{1}{\log x}\times\frac{1}{x}\right)=\frac{1}{(x\log x)}.$$

$$\therefore \frac{d}{dx}\{\log(\log x)\}=\frac{1}{(x\log x)}.$$

EXAMPLE 3 *If* $y=e^{\sqrt{\cot x}}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y=e^{\sqrt{\cot x}}$.

Putting $\cot x=t$ and $\sqrt{\cot x}=\sqrt{t}=u$, we get

$$y=e^u,\ u=\sqrt{t} \text{ and } t=\cot x$$

$$\Rightarrow \frac{dy}{du}=e^u,\ \frac{du}{dt}=\frac{1}{2}t^{-1/2}=\frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx}=-\text{cosec}^2x$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{dy}{du}\times\frac{du}{dt}\times\frac{dt}{dx}\right)$$

$$=\left\{e^u\cdot\frac{1}{2\sqrt{t}}\cdot(-\text{cosec}^2x)\right\}=e^{\sqrt{\cot x}}\cdot\frac{1}{2\sqrt{\cot x}}\cdot(-\text{cosec}^2x)$$

$$=\frac{(-\text{cosec}^2x)\ e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}.$$

EXAMPLE 4 *If* $y=\log\tan\frac{x}{2}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y=\log\tan\frac{x}{2}$.

Putting $\frac{x}{2}=t$ and $\tan\frac{x}{2}=\tan t=u$, we get

$$y=\log u,\ u=\tan t \text{ and } t=\frac{x}{2}$$

$$\Rightarrow \frac{dy}{du}=\frac{1}{u},\ \frac{du}{dt}=\sec^2t \text{ and } \frac{dt}{dx}=\frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{dy}{du}\times\frac{du}{dt}\times\frac{dt}{dx}\right)$$

$$= \left(\frac{1}{u} \times \sec^2 t \times \frac{1}{2}\right) = \left(\frac{1}{2\tan t} \times \sec^2 t\right) = \frac{1}{\sin 2t} = \frac{1}{\sin x} = \text{cosec } x.$$

Hence, $\dfrac{dy}{dx} = \text{cosec } x.$

EXAMPLE 5 *If* $y = \dfrac{1}{\log \cos x}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $y = (\log \cos x)^{-1}$.

Putting $\cos x = t$ and $\log \cos x = \log t = u$, we get

$$y = u^{-1} = \frac{1}{u}, \quad u = \log t \text{ and } t = \cos x$$

$$\Rightarrow \frac{dy}{du} = \frac{-1}{u^2}, \quad \frac{du}{dt} = \frac{1}{t} \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left\{\frac{-1}{u^2} \times \frac{1}{t} \times (-\sin x)\right\} = \left\{\frac{1}{(\log \cos x)^2} \cdot \frac{1}{\cos x} \cdot \sin x\right\}$$

$$= \frac{\tan x}{(\log \cos x)^2}.$$

EXAMPLE 6 *If* $y = \sqrt{e^{\sqrt{x}}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $\sqrt{x} = t$, $e^{\sqrt{x}} = e^t = u$, we get

$$y = \sqrt{u}, \quad u = e^t \text{ and } t = \sqrt{x}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dt} = e^t \text{ and } \frac{dt}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{2\sqrt{u}} \times e^t \times \frac{1}{2\sqrt{x}}\right) = \left\{\frac{1}{2\sqrt{u}} \times u \times \frac{1}{2\sqrt{x}}\right\} = \frac{\sqrt{u}}{4\sqrt{x}} = \frac{e^{\frac{1}{2}t}}{4\sqrt{x}}$$

$$= \frac{e^{\frac{1}{2}\sqrt{x}}}{4\sqrt{x}}.$$

EXAMPLE 7 *If* $y = \log \log \log x^3$, *find* $\dfrac{dy}{dx}$.

SOLUTION Let $x^3 = t$, $\log x^3 = \log t = u$ and $\log \log x^3 = \log u = v$.

Then, $y = \log v$, $v = \log u$, $u = \log t$ and $t = x^3$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$$

$$= \left(\frac{1}{v} \times \frac{1}{u} \times \frac{1}{t} \times 3x^2\right) = \frac{3x^2}{tuv}$$

$$= \frac{3x^2}{x^3(\log t)(\log u)} = \frac{3}{x(\log t)(\log \log t)}$$

$$= \frac{3}{x(\log x^3)(\log \log x^3)} = \frac{1}{x(\log x)(\log \log x^3)}.$$

EXAMPLE 8 *If* $y = \sqrt{\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\}}$, *find* $\frac{dy}{dx}$.

SOLUTION Put $\left(\frac{x^2}{3} - 1\right) = t$, $\sin\left(\frac{x^2}{3} - 1\right) = \sin t = u$

and $\log\left\{\sin\left(\frac{x^2}{3} - 1\right)\right\} = \log u = v.$

Then, $y = \sqrt{v}$, where $v = \log u$, $u = \sin t$ and $t = \left(\frac{x^2}{3} - 1\right)$.

$\therefore \quad \frac{dy}{dv} = \frac{1}{2}v^{-1/2} = \frac{1}{2\sqrt{v}}$; $\frac{dv}{du} = \frac{1}{u}$; $\frac{du}{dt} = \cos t$ and $\frac{dt}{dx} = \frac{2x}{3}$.

So, $\frac{dy}{dx} = \left(\frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right)$

$$= \left(\frac{1}{2\sqrt{v}} \cdot \frac{1}{u} \cdot \cos t \cdot \frac{2x}{3}\right)$$

$$= \frac{x}{3} \cdot \frac{\cos t}{u \cdot \sqrt{\log u}} = \frac{x}{3} \cdot \frac{\cos t}{\sin t \sqrt{\log \sin t}}$$

[$\because \quad v = \log u$ and $u = \sin t$]

$$= \frac{x \cot t}{3\sqrt{\log \sin t}} = \frac{x \cot\left(\frac{x^2}{3} - 1\right)}{3 \cdot \sqrt{\log \sin\left(\frac{x^2}{3} - 1\right)}} \qquad \left[\because \ t = \left(\frac{x^2}{3} - 1\right)\right].$$

EXAMPLE 9 *If* $y = e^x \log(\sin 2x)$, *find* $\frac{dy}{dx}$.

SOLUTION We have

$$\frac{dy}{dx} = \frac{d}{dx}\{e^x \log(\sin 2x)\}$$

$$= e^x \cdot \frac{d}{dx}\{\log(\sin 2x)\} + \log(\sin 2x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \left\{\frac{1}{\sin 2x} \cdot \cos 2x \cdot 2\right\} + \log(\sin 2x) \cdot e^x$$

$$= 2e^x \cot 2x + e^x \log(\sin 2x)$$

$$= e^x\{2\cot 2x + \log(\sin 2x)\}.$$

EXAMPLE 10 *If* $y = e^{ax}\cos(bx + c)$, *find* $\frac{dy}{dx}$.

SOLUTION We have

$$\frac{dy}{dx} = e^{ax}\cdot\frac{d}{dx}\{\cos(bx+c)\} + \cos(bx+c)\cdot\frac{d}{dx}(e^{ax})$$

$$= e^{ax}\cdot\{-b\sin(bx+c)\} + \cos(bx+c)\cdot ae^{ax}$$

$$= e^{ax}\cdot\{a\cos(bx+c) - b\sin(bx+c)\}.$$

EXAMPLE 11 *Differentiate* $\log\sqrt{\frac{1+\cos^2 x}{(1-e^{2x})}}$ *w.r.t. x.* **[CBSE 2003C]**

SOLUTION Let $y = \log\sqrt{\frac{1+\cos^2 x}{(1-e^{2x})}} = \log\left(\frac{1+\cos^2 x}{1-e^{2x}}\right)^{1/2} = \frac{1}{2}\log\left(\frac{1+\cos^2 x}{1-e^{2x}}\right)$.

$$\therefore\quad y = \frac{1}{2}\log(1+\cos^2 x) - \frac{1}{2}\log(1-e^{2x})$$

$$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{(1+\cos^2 x)}(2\cos x)(-\sin x) - \frac{1}{2}\cdot\frac{1}{(1-e^{2x})}\cdot(-2e^{2x})$$

$$= \left\{\frac{-\sin x\cos x}{(1+\cos^2 x)} + \frac{e^{2x}}{(1-e^{2x})}\right\}.$$

Hence, $\frac{d}{dx}\left\{\log\sqrt{\frac{1+\cos^2 x}{(1-e^{2x})}}\right\} = \left\{\frac{-\sin x\cos x}{(1+\cos^2 x)} + \frac{e^{2x}}{(1-e^{2x})}\right\}$.

EXAMPLE 12 *If* $y = \log\sqrt{\frac{1+\sin^2 x}{1-\sin x}}$, *find* $\frac{dy}{dx}$. **[CBSE 2003C]**

SOLUTION We have

$$y = \frac{1}{2}\{\log(1+\sin^2 x) - \log(1-\sin x)\}. \qquad \text{... (i)}$$

On differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\cdot\left[\frac{d}{dx}\{\log(1+\sin^2 x)\} - \frac{d}{dx}\{\log(1-\sin x)\}\right]$$

$$= \frac{1}{2}\cdot\left\{\frac{2\sin x\cos x}{(1+\sin^2 x)} - \frac{(-\cos x)}{(1-\sin x)}\right\}$$

$$= \frac{1}{2}\cdot\left\{\frac{\sin 2x}{(1+\sin^2 x)} + \frac{\cos x}{(1-\sin x)}\right\}.$$

EXAMPLE 13 *If* $y = \log\sin(e^x + 5x + 8)$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = \log\sin(e^x + 5x + 8)$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sin(e^x + 5x + 8)} \cdot \cos(e^x + 5x + 8) \cdot \frac{d}{dx}(e^x + 5x + 8)$$

$$= \{\cot(e^x + 5x + 8)\}(e^x + 5) = (e^x + 5) \cdot \cot(e^x + 5x + 8).$$

EXAMPLE 14 *If* $y = \sqrt{x^2+1} - \log\left\{\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right\}$, *find* $\frac{dy}{dx}$. **[CBSE 2008]**

SOLUTION We have

$$y = \sqrt{x^2+1} - \log\left\{\frac{1+\sqrt{x^2+1}}{x}\right\}$$

$$\Rightarrow \quad y = \sqrt{x^2+1} - \log\{1+\sqrt{x^2+1}\} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - \frac{1}{\{1+\sqrt{x^2+1}\}} \cdot \left\{\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right\} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{1}{\{1+\sqrt{x^2+1}\}} \cdot \frac{x}{\sqrt{x^2+1}} + \frac{1}{x}$$

$$= \frac{x\{1+\sqrt{x^2+1}\} - x}{(\sqrt{x^2+1})\{1+\sqrt{x^2+1}\}} + \frac{1}{x} = \frac{x\sqrt{x^2+1}}{(\sqrt{x^2+1})\{1+\sqrt{x^2+1}\}} + \frac{1}{x}$$

$$= \frac{x}{\{1+\sqrt{x^2+1}\}} + \frac{1}{x} = \frac{(x^2+1)+\sqrt{x^2+1}}{x\{1+\sqrt{x^2+1}\}}$$

$$= \frac{(\sqrt{x^2+1})\{(\sqrt{x^2+1})+1\}}{x\{1+\sqrt{x^2+1}\}} = \frac{\sqrt{x^2+1}}{x}.$$

EXERCISE 10B

Differentiate each of the following w.r.t. x:

1. (i) e^{4x} (ii) e^{-5x} (iii) e^{x^3}
2. (i) $e^{2/x}$ (ii) $e^{\sqrt{x}}$ (iii) $e^{-2\sqrt{x}}$
3. (i) $e^{\cot x}$ (ii) $e^{-\sin 2x}$ (iii) $e^{\sqrt{\sin x}}$
4. (i) $\tan(\log x)$ (ii) $\log \sec x$ (iii) $\log \sin \frac{x}{2}$
5. (i) $\log_3 x$ (ii) 2^{-x} (iii) 3^{x+2}
6. (i) $\log\left(x + \frac{1}{x}\right)$ (ii) $\log \sin 3x$ (iii) $\log(x + \sqrt{1+x^2})$ **[CBSE 2003]**
7. $e^{\sqrt{x}} \log x$
8. $\log \sin \sqrt{x^2+1}$ **[CBSE 2003]**
9. $e^{2x} \sin 3x$
10. $e^{3x} \cos 2x$
11. $e^{-5x} \cot 4x$
12. $e^x \log(\sin 2x)$
13. $\log(\operatorname{cosec} x - \cot x)$
14. $\log\left(\sec \frac{x}{2} + \tan \frac{x}{2}\right)$

15. $\sqrt{\dfrac{1+e^x}{1-e^x}}$
16. $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$
17. $xe^{\sqrt{\sin x}}$
18. $e^{\sin x}\sin(e^x)$
19. $e^{\sqrt{1-x^2}}\tan x$
20. $\dfrac{e^x}{(1+\cos x)}$
21. $x^3e^x\cos x$
22. $e^{x\cos x}$

ANSWERS (EXERCISE 10B)

1. (i) $4e^{4x}$ (ii) $-5e^{-5x}$ (iii) $(3x^2)e^{x^3}$

2. (i) $-\dfrac{2}{x^2}\cdot e^{2/x}$ (ii) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$ (iii) $\dfrac{-1}{\sqrt{x}}e^{-2\sqrt{x}}$

3. (i) $-e^{\cot x}(\text{cosec}^2x)$ (ii) $(-2\cos 2x)e^{-\sin 2x}$ (iii) $\dfrac{\cos x}{2\sqrt{\sin x}}\cdot e^{\sqrt{\sin x}}$

4. (i) $\dfrac{\sec^2(\log x)}{x}$ (ii) $\tan x$ (iii) $\dfrac{1}{2}\cot\dfrac{x}{2}$

5. (i) $\dfrac{1}{x(\log 3)}$ (ii) $-2^{-x}\log 2$ (iii) $(9\times 3^x\log 3)$

6. (i) $\dfrac{(x^2-1)}{x(x^2+1)}$ (ii) $3\cot 3x$ (iii) $\dfrac{1}{\sqrt{1+x^2}}$

7. $\dfrac{e^{\sqrt{x}}(2+\sqrt{x}\log x)}{2x}$ **8.** $\dfrac{x}{\sqrt{x^2+1}}\cot\sqrt{x^2+1}$ **9.** $e^{2x}(3\cos 3x+2\sin 3x)$

10. $e^{3x}(3\cos 2x-2\sin 2x)$ **11.** $-e^{-5x}(5\cot 4x+4\text{cosec}^2 4x)$

12. $e^x\{2\cot 2x+\log\sin 2x\}$ **13.** $\text{cosec}\,x$ **14.** $\dfrac{1}{2}\sec\dfrac{x}{2}$

15. $\dfrac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$ **16.** $\dfrac{-4}{(e^x-e^{-x})^2}$

17. $e^{\sqrt{\sin x}}\cdot\left(\dfrac{x\cos x}{2\sqrt{\sin x}}+1\right)$ **18.** $e^{\sin x}(e^x\cos e^x+\cos x\sin e^x)$

19. $e^{\sqrt{1-x^2}}\left\{\sec^2x-\dfrac{x\tan x}{\sqrt{1-x^2}}\right\}$ **20.** $\dfrac{e^x(1+\cos x+\sin x)}{(1+\cos x)^2}$

21. $e^xx^2(x\cos x-x\sin x+3\cos x)$ **22.** $e^{x\cos x}(\cos x-x\sin x)$

3. Derivatives of Inverse Trigonometric Functions

In the table given below, we mention the domain and range of various inverse trigonometric functions.

Function	Domain	Range
(i) $\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
(iii) $\tan^{-1}x$	R	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(iv) $\cot^{-1}x$	R	$]0, \pi[$
(v) $\sec^{-1}x$	$R -]-1, 1[$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
(vi) $\text{cosec}^{-1}x$	$R -]-1, 1[$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

Derivatives of Inverse Trigonometric Functions

EXAMPLE 1 *Prove that* $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, *where* $x \in]-1, 1[$.

SOLUTION Let $y = \sin^{-1}x$, where $x \in]-1, 1[$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Then,

$$y = \sin^{-1}x \Rightarrow x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y \geq 0 \text{ since } y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\Rightarrow \frac{dx}{dy} = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Hence, $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

EXAMPLE 2 *Prove that* $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$, *where* $x \in]-1, 1[$.

SOLUTION Let $y = \cos^{-1}x$, where $x \in]-1, 1[$ and $y \in \left]0, \frac{\pi}{2}\right[$. Then,

$$y = \cos^{-1}x \Rightarrow x = \cos y$$

$$\Rightarrow \frac{dx}{dy} = -\sin y, \text{ where } \sin y > 0, \text{ since } y \in \left]0, \frac{\pi}{2}\right[$$

$$\Rightarrow \frac{dx}{dy} = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

Hence, $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$.

EXAMPLE 3 *Prove that* $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$, *where* $x \in R$.

SOLUTION Let $y = \tan^{-1}x$, where $x \in R$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Then,

$$x = \tan y$$
$$\Rightarrow \frac{dx}{dy} = \sec^2 y = (1 + \tan^2 y) = (1 + x^2)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1+x^2)}.$$

Hence, $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$.

EXAMPLE 4 *Prove that* $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$, *where* $x \in R$.

SOLUTION Let $y = \cot^{-1}x$, where $x \in R$ and $y \in]0, \pi[$. Then,

$$x = \cot y$$
$$\Rightarrow \frac{dx}{dy} = -\text{cosec}^2 y = -(1 + \cot^2 y) = -(1 + x^2)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+x^2)}.$$

Hence, $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$.

EXAMPLE 5 *Prove that* $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$, *where* $x \in R - [-1, 1]$.

SOLUTION Let $y = \sec^{-1}x$, where $x \in R - [-1, 1]$ and $y \in]0, \pi[- \left\{\frac{\pi}{2}\right\}$. Then,

$$x = \sec y$$
$$\Rightarrow \frac{dx}{dy} = \sec y \tan y > 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \cdot \sqrt{\sec^2 y - 1}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}.$$

Hence, $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$.

EXAMPLE 6 *Prove that* $\dfrac{d}{dx}(\text{cosec}^{-1}x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$, *where* $x \in R - [-1, 1]$.

SOLUTION Let $y = \text{cosec}^{-1}x$, where $x \in R - [-1, 1]$ and $y \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[- \{0\}$. Then,

$$x = \text{cosec}\, y$$

$$\Rightarrow \frac{dx}{dy} = -\text{cosec}\, y \cot y, \text{ where cosec } y \cot y > 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec}\, y \cot y} = \frac{-1}{(\text{cosec}\, y)\sqrt{\text{cosec}^2 y - 1}} = \frac{-1}{|x|\sqrt{x^2-1}}.$$

Hence, $\dfrac{d}{dx}(\text{cosec}^{-1}x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

SUMMARY

(i) $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$ **(ii)** $\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$

(iii) $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{(1+x^2)}$ **(iv)** $\dfrac{d}{dx}(\cot^{-1}x) = \dfrac{-1}{(1+x^2)}$

(v) $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{|x|\sqrt{x^2-1}}$ **(vi)** $\dfrac{d}{dx}(\text{cosec}^{-1}x) = \dfrac{-1}{|x|\cdot\sqrt{x^2-1}}$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate the following w.r.t. x:*
(i) $\sin^{-1}2x$ *(ii)* $\tan^{-1}\sqrt{x}$ *(iii)* $\cos^{-1}(\cot x)$

SOLUTION (i) Let $y = \sin^{-1}2x$.

Putting $2x = t$, we get $y = \sin^{-1}t$ and $t = 2x$.

Now, $y = \sin^{-1}t \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{\sqrt{1-t^2}}$.

And, $t = 2x \Rightarrow \dfrac{dt}{dx} = 2$.

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{2}{\sqrt{1-t^2}} = \frac{2}{\sqrt{1-4x^2}} \qquad [\because \quad t = 2x].$$

Hence, $\dfrac{d}{dx}(\sin^{-1}2x) = \dfrac{2}{\sqrt{1-4x^2}}$.

(ii) Let $y = \tan^{-1}\sqrt{x}$.

Putting $\sqrt{x} = t$, we get $y = \tan^{-1}t$ and $t = \sqrt{x}$.

Now, $y = \tan^{-1}t \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{(1+t^2)}$.

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{1}{(1+t^2)} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$ $[\because t = \sqrt{x}]$.

Hence, $\dfrac{d}{dx}(\tan^{-1}\sqrt{x}) = \dfrac{1}{2\sqrt{x}(1+x)}$.

(iii) Let $y = \cos^{-1}(\cot x)$.

Putting $\cot x = t$, we get $y = \cos^{-1}t$ and $t = \cot x$.

Now, $y = \cos^{-1}t \Rightarrow \dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}}$.

And, $t = \cot x \Rightarrow \dfrac{dt}{dx} = -\text{cosec}^2 x$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{\text{cosec}^2 x}{\sqrt{1-t^2}} = \dfrac{\text{cosec}^2 x}{\sqrt{1-\cot^2 x}}$ $[\because t = \cot x]$.

Hence, $\dfrac{d}{dx}\{\cos^{-1}(\cot x)\} = \dfrac{\text{cosec}^2 x}{\sqrt{1-\cot^2 x}}$.

EXAMPLE 2 *Differentiate the following w.r.t. x:*

(i) $\sec(\tan^{-1}x)$ *(ii)* $\sin(\tan^{-1}x)$ *(iii)* $\cot(\cos^{-1}x)$

SOLUTION (i) Let $y = \sec(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sec t$ and $t = \tan^{-1}x$.

Now, $y = \sec t \Rightarrow \dfrac{dy}{dt} = \sec t \tan t$.

And, $t = \tan^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{(1+x^2)}$.

$\therefore \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{\sec t \tan t}{(1+x^2)} = \dfrac{(\sqrt{1+\tan^2 t})(\tan t)}{(1+x^2)}$

$= \dfrac{(\sqrt{1+x^2})x}{(1+x^2)} = \dfrac{x}{\sqrt{1+x^2}}$ $[\because t = \tan^{-1}x \Rightarrow \tan t = x]$

.

Hence, $\dfrac{d}{dx}\{\sec(\tan^{-1}x)\} = \dfrac{x}{\sqrt{1+x^2}}$.

(ii) Let $y = \sin(\tan^{-1}x)$.

Putting $\tan^{-1}x = t$, we get $y = \sin t$ and $t = \tan^{-1}x$.

Now, $y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$.

And, $t = \tan^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{(1+x^2)}$.

$\therefore \quad \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \cos t \cdot \dfrac{1}{(1+x^2)} = \dfrac{1}{(1+x^2)^{3/2}}$

$\left[\because \tan t = x \Rightarrow \cos t = \dfrac{1}{\sqrt{1+x^2}}\right]$.

Hence, $\dfrac{d}{dx}\{\sin(\tan^{-1}x)\} = \dfrac{1}{(1+x^2)^{3/2}}$.

(iii) Let $y = \cot(\cos^{-1}x)$.

Putting $\cos^{-1}x = t$, we get $y = \cot t$ and $t = \cos^{-1}x$.

Now, $y = \cot t \Rightarrow \dfrac{dy}{dt} = -\text{cosec}^2 t$.

And, $t = \cos^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$.

$\therefore \quad \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{\text{cosec}^2 t}{\sqrt{1-x^2}} = \dfrac{1}{(1-x^2)^{3/2}}$

$\left[\because \cos t = x \Rightarrow \text{cosec}^2 t = \dfrac{1}{(1-x^2)}\right]$.

Hence, $\dfrac{d}{dx}\{\cot(\cos^{-1}x)\} = \dfrac{1}{(1-x^2)^{3/2}}$.

EXAMPLE 3 *If* $y = \sin(\tan^{-1}2x)$, *prove that* $\dfrac{dy}{dx} = \dfrac{2}{(1+4x^2)^{3/2}}$.

SOLUTION Putting $\tan^{-1}2x = t$, we get $y = \sin t$ and $t = \tan^{-1}2x$.

Now, $y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t$.

And, $t = \tan^{-1}2x \Rightarrow \dfrac{dt}{dx} = \left\{\dfrac{1}{(1+4x^2)} \times 2\right\} = \dfrac{2}{(1+4x^2)}$.

$\therefore \quad \dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \left\{\cos t \times \dfrac{2}{(1+4x^2)}\right\}$. ... (i)

Now, $t = \tan^{-1}2x \Rightarrow \tan t = 2x$

$\Rightarrow \sec t = \sqrt{1+\tan^2 t} = \sqrt{1+4x^2}$

$\Rightarrow \cos t = \dfrac{1}{\sec t} = \dfrac{1}{\sqrt{1+4x^2}}$. ... (ii)

Putting the value of $\cos t$ from (ii) in (i), we get

$\dfrac{dy}{dx} = \left\{\dfrac{1}{\sqrt{1+4x^2}} \times \dfrac{2}{(1+4x^2)}\right\} = \dfrac{2}{(1+4x^2)^{3/2}}$.

EXAMPLE 4 *Differentiate* $\sqrt{\cot^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{\cot^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\cot^{-1}\sqrt{x} = \cot^{-1}t = u$, we get

$$y = \sqrt{u}, \text{ where } u = \cot^{-1}t \text{ and } t = \sqrt{x}.$$

Now, $y = \sqrt{u} \Rightarrow \dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$;

$$u = \cot^{-1}t \Rightarrow \frac{du}{dt} = \frac{-1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = \frac{-1}{4\sqrt{u}(1+t^2)\sqrt{x}}$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}t})(1+t^2)\sqrt{x}} \qquad [\because \quad u = \cot^{-1}t]$$

$$= \frac{-1}{4(\sqrt{\cot^{-1}\sqrt{x}})(1+x)\sqrt{x}} \qquad [\because \quad t = \sqrt{x}].$$

EXAMPLE 5 *Differentiate* $e^{\tan^{-1}\sqrt{x}}$ *w.r.t. x.*

SOLUTION Let $y = e^{\tan^{-1}\sqrt{x}}$.

Putting $\sqrt{x} = t$ and $\tan^{-1}\sqrt{x} = \tan^{-1}t = u$, we get

$$y = e^u, \text{ where } u = \tan^{-1}t \text{ and } t = \sqrt{x}.$$

Now, $y = e^u \Rightarrow \dfrac{dy}{du} = e^u$;

$$u = \tan^{-1}t \Rightarrow \frac{du}{dt} = \frac{1}{(1+t^2)}.$$

And, $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}\right) = e^u \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}}$$

$$= e^{\tan^{-1}t} \cdot \frac{1}{(1+t^2)} \cdot \frac{1}{2\sqrt{x}} \qquad [\because \quad u = \tan^{-1}t]$$

$$= \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)} \qquad [\because \quad t = \sqrt{x}].$$

Hence, $\dfrac{dy}{dx} = \dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$.

EXAMPLE 6 *If* $y = \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION $$y = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \quad \text{... (i)}$$

$$\Rightarrow \; y\sqrt{1-x^2} = x \sin^{-1} x. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\sqrt{1-x^2}) + (\sqrt{1-x^2})\frac{dy}{dx} = x \cdot \frac{d}{dx}(\sin^{-1} x) + \sin^{-1} x \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \; y \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) + (\sqrt{1-x^2})\frac{dy}{dx} = x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot 1$$

$$\Rightarrow \; \frac{-x\,y}{\sqrt{1-x^2}} + (\sqrt{1-x^2})\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$$

$$\Rightarrow \; \frac{-x^2 \sin^{-1} x}{(1-x^2)} + (\sqrt{1-x^2})\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x \qquad \text{[using (i)]}$$

$$\Rightarrow \; -x^2 \sin^{-1} x + (1-x^2)^{3/2}\frac{dy}{dx} = x(\sqrt{1-x^2}) + (1-x^2)\sin^{-1} x$$

$$\Rightarrow \; (1-x^2)^{3/2}\frac{dy}{dx} = x(\sqrt{1-x^2}) + \sin^{-1} x$$

$$\Rightarrow \; \frac{dy}{dx} = \frac{x(\sqrt{1-x^2}) + \sin^{-1} x}{(1-x^2)^{3/2}}.$$

EXAMPLE 7 *Find* $\dfrac{d}{dx}[\sqrt{1-x^2} \sin^{-1} x - x]$.

SOLUTION We have

$$\frac{d}{dx}[(\sqrt{1-x^2})\sin^{-1} x - x]$$

$$= \frac{d}{dx}[(\sqrt{1-x^2}\sin^{-1} x] - \frac{d}{dx}(x)$$

$$= (\sqrt{1-x^2}) \cdot \frac{d}{dx}(\sin^{-1} x) + (\sin^{-1} x) \cdot \frac{d}{dx}(\sqrt{1-x^2}) - 1$$

$$= (\sqrt{1-x^2}) \cdot \frac{1}{(\sqrt{1-x^2})} + (\sin^{-1} x) \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) - 1$$

$$= \left\{1 - \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - 1\right\} = \frac{-x \sin^{-1} x}{\sqrt{1-x^2}}.$$

EXAMPLE 8 *Show that* $\dfrac{d}{dx}\left[\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}\right]=\sqrt{a^2-x^2}.$ **[CBSE 2004C]**

SOLUTION We have

$$\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}\right]+\frac{a^2}{2}\cdot\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{x}{2}\cdot\frac{d}{dx}(\sqrt{a^2-x^2})+(\sqrt{a^2-x^2})\cdot\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\dfrac{x^2}{a^2}}}\cdot\frac{1}{a}$$

$$=\frac{x}{2}\cdot\frac{1}{2}(a^2-x^2)^{-1/2}\cdot(-2x)+(\sqrt{a^2-x^2})\cdot\frac{1}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2+(a^2-x^2)+a^2}{2\sqrt{a^2-x^2}}=\frac{(a^2-x^2)}{\sqrt{(a^2-x^2)}}=\sqrt{(a^2-x^2)}.$$

Hence, $\dfrac{d}{dx}\left[\dfrac{x}{2}\cdot\sqrt{a^2-x^2}+\dfrac{a^2}{2}\cdot\sin^{-1}\dfrac{x}{a}\right]=\sqrt{a^2-x^2}.$

EXERCISE 10C

Differentiate each of the following w.r.t. x:

1. $\cos^{-1}2x$
2. $\tan^{-1}x^2$
3. $\sec^{-1}\sqrt{x}$
4. $\sin^{-1}\dfrac{x}{a}$
5. $\tan^{-1}(\log\ x)$
6. $\cot^{-1}(e^x)$
7. $\log\ (\tan^{-1}x)$
8. $\cot^{-1}x^3$
9. $\sin^{-1}(\cos\ x)$
10. $(1+x^2)\tan^{-1}x$
11. $\tan^{-1}(\cot\ x)$
12. $\log\ (\sin^{-1}x^4)$
13. $(\cot^{-1}x^2)^3$
14. $\tan^{-1}(\cos\sqrt{x})$
15. $\tan\ (\sin^{-1}x)$
16. $e^{\tan^{-1}\sqrt{x}}$
17. $\sqrt{\sin^{-1}x^2}$
18. If $y=\sin^{-1}(\cos\ x)+\cos^{-1}(\sin\ x)$, prove that $\dfrac{dy}{dx}=-2$.
19. Prove that $\dfrac{d}{dx}\{2x\tan^{-1}x-\log\ (1+x^2)\}=2\tan^{-1}x.$

ANSWERS (EXERCISE 10C)

1. $\dfrac{-2}{\sqrt{1-4x^2}}$ **2.** $\dfrac{2x}{(1+x^4)}$ **3.** $\dfrac{1}{2x\sqrt{x-1}}$ **4.** $\dfrac{1}{\sqrt{a^2-x^2}}$

5. $\dfrac{1}{x\{1+(\log\ x)^2\}}$ **6.** $\dfrac{-e^x}{(1+e^{2x})}$ **7.** $\dfrac{1}{(1+x^2)\tan^{-1}x}$ **8.** $\dfrac{-3x^2}{(1+x^6)}$

9. -1 **10.** $(1+2x\tan^{-1}x)$ **11.** -1 **12.** $\dfrac{4x^3}{(\sin^{-1}x^4)\sqrt{1-x^8}}$

13. $\dfrac{-6x(\cot^{-1}x^2)^2}{(1+x^4)}$ **14.** $\dfrac{-\sin\sqrt{x}}{(2\sqrt{x})(1+\cos^2\sqrt{x})}$ **15.** $\dfrac{1}{(1-x^2)^{3/2}}$

16. $\dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$ **17.** $\dfrac{x}{(\sqrt{1-x^4})\sqrt{\sin^{-1}x^2}}$

4. Differentiation by Trigonometrical Transformations

SOME USEFUL RESULTS

(i) $(1-\cos x)=2\sin^2\left(\dfrac{x}{2}\right)$ **(ii)** $(1+\cos x)=2\cos^2\left(\dfrac{x}{2}\right)$

(iii) $\sin 3x=(3\sin x-4\sin^3 x)$ **(iv)** $\cos 3x=(4\cos^3 x-3\cos x)$

(v) $\sin x=\dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$ **(vi)** $\cos x=\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

(vii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left\{\dfrac{x-y}{1+xy}\right\}$ **(viii)** $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left\{\dfrac{x+y}{1-xy}\right\}$

SOME USEFUL SUBSTITUTIONS

Suppose we are given $\sin^{-1}f(x)$, $\cos^{-1}f(x)$, $\tan^{-1}f(x)$, etc.

Rule 1. If $f(x)=\sqrt{a^2-x^2}$, put $x=a\sin\theta$ or $x=a\cos\theta$.

Rule 2. If $f(x)=\sqrt{a^2+x^2}$, put $x=a\tan\theta$ or $x=a\cot\theta$.

Rule 3. If $f(x)=\sqrt{x^2-a^2}$, put $x=a\sec\theta$ or $x=a\operatorname{cosec}\theta$.

Rule 4. If $f(x)=\sqrt{a-x}$, put $x=a\cos 2\theta$.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate each of the following w.r.t. x:*

(i) $\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ *(ii)* $\tan^{-1}\left(\dfrac{\cos x-\sin x}{\cos x+\sin x}\right)$

SOLUTION (i) Let $y=\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)=\tan^{-1}\left\{\dfrac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\}$

$=\tan^{-1}\left\{\tan\dfrac{x}{2}\right\}=\dfrac{x}{2}.$

$$\therefore \quad y = \frac{x}{2}.$$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}.$

(ii) Let $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$

[on dividing num. and denom. by $\cos x$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - x\right)\right\} = \left(\frac{\pi}{4} - x\right).$$

$$\therefore \quad y = \left(\frac{\pi}{4} - x\right).$$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}(x) = (0 - 1) = -1.$

EXAMPLE 2 *Differentiate w.r.t. x:*

(i) $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$ *(ii)* $\tan^{-1}(\sec x + \tan x)$

SOLUTION (i) Let $y = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\}$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

$$\therefore \quad y = \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0 - \frac{1}{2}\right) = \frac{-1}{2}.$

(ii) Let $y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$$

$$= \tan^{-1}\left\{\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right\}$$

$$\left\{\because \quad \cos\left(\frac{\pi}{2} + x\right) = -\sin x; \ \sin\left(\frac{\pi}{2} + x\right) = \cos x\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\therefore \quad y = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

EXAMPLE 3 *Differentiate w.r.t. x:*

(i) $\tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\}$ *(ii)* $\tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$ **[CBSE 2003C, '04C]**

SOLUTION (i) Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\cos^2(x/2)}{2\sin^2(x/2)}}\right\}$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\} = \left(\frac{\pi}{2}-\frac{x}{2}\right).$$

$$\therefore \quad y = \left(\frac{\pi}{2}-\frac{x}{2}\right).$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{d}{dx}\left(\frac{x}{2}\right) = \left(0-\frac{1}{2}\right) = \frac{-1}{2}.$$

(ii) Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\} = \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{1+\cos\left(\frac{\pi}{2}+x\right)}\right\}^{\frac{1}{2}}$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}^{\frac{1}{2}} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\therefore \quad y = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\text{Hence, } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

EXAMPLE 4 *Differentiate* $\cos^{-1}\left\{\sqrt{\frac{1+\cos x}{2}}\right\}$ *w.r.t. x.*

SOLUTION Let $y = \cos^{-1}\left\{\sqrt{\frac{1+\cos x}{2}}\right\} = \cos^{-1}\left\{\sqrt{\frac{2\cos^2(x/2)}{2}}\right\}$

$= \cos^{-1}\{\cos(x/2)\} = \frac{x}{2}$.

$\therefore \quad y = \frac{x}{2}$.

Hence, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$.

EXAMPLE 5 *If* $y = \cot^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}$, *find* $\frac{dy}{dx}$. **[CBSE 2004C]**

SOLUTION We have

$$y = \cot^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}} = \cot^{-1}\sqrt{\frac{1+\cos\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}}$$

$$= \cot^{-1}\sqrt{\frac{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}} = \cot^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

EXAMPLE 6 *If* $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$ *and* $0 < x < \frac{\pi}{2}$, *find* $\frac{dy}{dx}$. **[CBSE 2008]**

SOLUTION We have

$$(1+\sin x) = \{\cos^2(x/2) + \sin^2(x/2) + 2\sin(x/2)\cos(x/2)\}$$
$$= \{\cos(x/2) + \sin(x/2)\}^2.$$
$$(1-\sin x) = \{\cos^2(x/2) + \sin^2(x/2) - 2\sin(x/2)\cos(x/2)\}$$
$$= \{\cos(x/2) - \sin(x/2)\}^2.$$

$$\therefore \quad \sqrt{1+\sin x} = \sqrt{\{\cos(x/2) + \sin(x/2)\}^2} = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)$$

and $\sqrt{1-\sin x} = \sqrt{\{\cos(x/2) - \sin(x/2)\}^2} = \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)$.

$$\therefore \quad y = \cot^{-1}\left\{\frac{[\cos(x/2) + \sin(x/2)] + [\cos(x/2) - \sin(x/2)]}{[\cos(x/2) + \sin(x/2)] - [\cos(x/2) - \sin(x/2)]}\right\}$$

$$= \cot^{-1}\left\{\frac{2\cos(x/2)}{2\sin(x/2)}\right\} = \cot^{-1}\{\cot(x/2)\} = \frac{x}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}.$$

EXAMPLE 7 *If* $y = \tan^{-1}\dfrac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2004]**

SOLUTION We have

$$y = \tan^{-1}\left\{\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} \times \frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x}}{(\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right\}$$

$$= \tan^{-1}\left\{\frac{(1+\sin x)+(1-\sin x)+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}\right\} = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\} = \tan^{-1}\left\{\cot\frac{x}{2}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\}$$

$$= \left(\frac{\pi}{2}-\frac{x}{2}\right).$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}\left(\frac{x}{2}\right) = \left(0-\frac{1}{2}\right) = -\frac{1}{2}.$$

EXAMPLE 8 *Differentiate w.r.t. x:*

(i) $\cot^{-1}\left(\dfrac{1}{x}\right)$ *(ii)* $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ *(iii)* $\cot^{-1}\left(\dfrac{1-x}{1+x}\right)$

SOLUTION (i) Let $y = \cot^{-1}\left(\dfrac{1}{x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1}{\tan\theta}\right) = \cot^{-1}(\cot\theta) = \theta = \tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(1+x^2)}.$$

Hence, $\dfrac{d}{dx}\left\{\cot^{-1}\left(\dfrac{1}{x}\right)\right\} = \dfrac{1}{(1+x^2)}$.

(ii) Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}.$$

Hence, $\dfrac{d}{dx}\left\{\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right\} = \dfrac{2}{(1+x^2)}$.

(iii) Let $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1-\tan\theta}{1+\tan 0}\right) = \cot^{-1}\left\{\tan\left(\frac{\pi}{4}-0\right)\right\}$$

$$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-0\right)\right\}\right] = \left(\frac{\pi}{4}+0\right) = \frac{\pi}{4}+\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{(1+x^2)}.$$

Hence, $\frac{d}{dx}\left\{\cot^{-1}\left(\frac{1-x}{1+x}\right)\right\} = \frac{1}{(1+x^2)}$.

EXAMPLE 9 *Differentiate w.r.t. x:*

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *(ii)* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *(iii)* $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

SOLUTION (i) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2 0}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}.$$

Hence, $\frac{d}{dx}\left\{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\} = \frac{2}{(1+x^2)}$.

(ii) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan 0}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(1+x^2)}.$$

Hence, $\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} = \frac{2}{(1+x^2)}$.

(iii) Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

Putting $x = \cos\theta$, we get

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x.$$

$\therefore \quad y = 2\cos^{-1}x.$

Hence, $\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-2}{\sqrt{1-x^2}}.$

EXAMPLE 10 *Differentiate w.r.t. x:*

(i) $\cos^{-1}(4x^3 - 3x)$ *(ii)* $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *(iii)* $\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$

SOLUTION (i) Let $y = \cos^{-1}(4x^3 - 3x).$

Putting $x = \cos\theta$, we get

$y = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = 3\theta.$

$\therefore \quad y = 3\theta \Rightarrow y = 3\cos^{-1}x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-3}{\sqrt{1-x^2}}.$

(ii) Let $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right).$

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right]$$

$$= \left(\frac{\pi}{2} - 2\theta\right) = \left(\frac{\pi}{2} - 2\tan^{-1}x\right).$$

$\therefore \quad y = \left(\dfrac{\pi}{2} - 2\tan^{-1}x\right).$

Hence, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 2\tan^{-1}x\right) = \dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) - 2\cdot\dfrac{d}{dx}(\tan^{-1}x)$

$$= \left\{0 - \frac{2}{(1+x^2)}\right\} = \frac{-2}{(1+x^2)}.$$

(iii) Let $y = \sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right).$

Putting $x = \cot\theta$, we get

$$y = \sec^{-1}\left(\frac{\cot^2\theta + 1}{\cot^2\theta - 1}\right) = \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cot^{-1}x.$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{-2}{(1+x^2)}.$

Hence, $\dfrac{d}{dx}\left\{\sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)\right\} = \dfrac{-2}{(1+x^2)}.$

EXAMPLE 11 *Differentiate* $\cot^{-1}\left(\frac{1-x}{1+x}\right)$ *w.r.t. x.* **[CBSE 2004]**

SOLUTION Let $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cot^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \cot^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$= \cot^{-1}\left[\cot\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right\}\right] = \cot^{-1}\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \left(\frac{\pi}{4}+\theta\right)$$

$$= \frac{\pi}{4} + \tan^{-1}x \qquad [\because \quad x = \tan\theta \Rightarrow \theta = \tan^{-1}x].$$

$$\therefore \quad y = \frac{\pi}{4} + \tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4} + \tan^{-1}x\right\} = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}(\tan^{-1}x)$$

$$= \left\{0 + \frac{1}{(1+x^2)}\right\} = \frac{1}{(1+x^2)}.$$

Hence, $\frac{dy}{dx} = \frac{1}{(1+x^2)}$.

EXAMPLE 12 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ *w.r.t. x.* **[CBSE 2004]**

SOLUTION Let $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$.

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\} = \tan^{-1}\left\{\frac{\sec\theta-1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\frac{1}{\cos\theta}-1\right)}{\sin\theta}\cdot\cos\theta\right\} = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad y = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}(\tan^{-1}x) = \frac{1}{2(1+x^2)}.$$

EXAMPLE 13 *If* $y = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2006]**

SOLUTION Putting $x^2 = \cos\theta$, we get

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 0}-\sqrt{1-\cos 0}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(\theta/2)}-\sqrt{2\sin^2(\theta/2)}}{\sqrt{2\cos^2(0/2)}+\sqrt{2}\sin^2(0/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2}\cos(\theta/2)-\sqrt{2}\sin(\theta/2)}{\sqrt{2}\cos(\theta/2)+\sqrt{2}\sin(\theta/2)}\right\} = \tan^{-1}\left\{\frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$= \left(\frac{\pi}{4}-\frac{0}{2}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2\right\} = \frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{1}{2}\cdot\frac{d}{dx}(\cos^{-1}x^2)$$

$$= \left\{0-\frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot 2x\right\} = \frac{x}{\sqrt{1-x^4}}.$$

EXAMPLE 14 *If* $y = \sin^{-1}[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}]$, *find* $\dfrac{dy}{dx}$. **[CBSE 2010]**

SOLUTION Putting $x = \sin\theta$ and $\sqrt{x} = \sin\phi$, we get

$$y = \sin^{-1}[\sin 0\cos\phi - \sin\phi\cos 0] = \sin^{-1}[\sin(0-\phi)]$$

$$= (0-\phi) = \sin^{-1}x - \sin^{-1}\sqrt{x}.$$

$$\therefore \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1}x - \sin^{-1}\sqrt{x}\}$$

$$= \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}\{\sin^{-1}\sqrt{x}\}$$

$$= \left[\frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\cdot\sqrt{1-x}}\right].$$

EXAMPLE 15 *Differentiate* $\tan^{-1}\left(\dfrac{x^{1/3}+a^{1/3}}{1-x^{1/3}a^{1/3}}\right)$ *w.r.t. x.*

SOLUTION Let $y = \tan^{-1}\left(\dfrac{x^{1/3}+a^{1/3}}{1-x^{1/3}a^{1/3}}\right)$.

Putting $x^{1/3} = \tan\theta$ and $a^{1/3} = \tan\phi$, we get

$$y = \tan^{-1}\left(\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}\right) = \tan^{-1}[\tan(\theta+\phi)]$$

$$= (\theta+\phi) = \tan^{-1}(x^{1/3}) + \tan^{-1}(a^{1/3}).$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\{\tan^{-1}(x^{1/3}) + \tan^{-1}(a^{1/3})\}$$

$$= \frac{d}{dx}\{\tan^{-1}(x^{1/3})\} + \frac{d}{dx}\{\tan^{-1}(a^{1/3})\} = \frac{1}{(1+x^{2/3})}\cdot\frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3x^{2/3}(1+x^{2/3})} \qquad [\because \quad \tan^{-1}(a^{1/3}) = \text{constant}].$$

EXAMPLE 16 *Differentiate* $\tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ *w.r.t. x.*

SOLUTION Let $y = \tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$.

Putting $a = r\sin\theta$ and $b = r\cos\theta$, we get

$$y = \tan^{-1}\left\{\frac{r(\sin\theta\cos x - \cos\theta\sin x)}{r(\cos\theta\cos x + \sin\theta\sin x)}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin(\theta-x)}{\cos(\theta-x)}\right\} = \tan^{-1}\{\tan(\theta-x)\}$$

$$= \theta - x = \left(\tan^{-1}\frac{a}{b} - x\right) \quad \left[\because \quad \frac{a}{b} = \tan\theta \;\Rightarrow\; \theta = \tan^{-1}\frac{a}{b}\right].$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b} - x\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x) = -1 \qquad \left[\because \quad \tan^{-1}\frac{a}{b} = \text{constant}\right].$$

EXAMPLE 17 *If* $y = \sin^{-1}\left\{\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2}\right\}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $x = \cos 2\theta$, we get

$$y = \sin^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{2}\right\} = \sin^{-1}\left\{\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right\} = \sin^{-1}\left\{\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4} - \theta\right)\right\}$$

$$= \left(\frac{\pi}{4} - \theta\right) = \left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right)$$

$$\left[\because\ x = \cos 2\theta \ \Rightarrow\ 2\theta = \cos^{-1}x \ \Rightarrow\ \theta = \frac{1}{2}\cos^{-1}x\right]$$

$$\therefore\ y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x.$$

Hence, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left\{\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x\right\} = \dfrac{d}{dx}\left(\dfrac{\pi}{4}\right) - \dfrac{1}{2}\dfrac{d}{dx}(\cos^{-1}x)$

$$= \left\{0 - \frac{1}{2}\cdot\frac{(-1)}{\sqrt{1-x^2}}\right\} = \frac{1}{2\sqrt{1-x^2}}.$$

EXAMPLE 18 *Differentiate each of the following w.r.t. x:*

(i) $\tan^{-1}(\sqrt{1+x^2}+x)$ *(ii)* $\tan^{-1}(\sqrt{1+x^2}-x)$

SOLUTION (i) Let $y = \tan^{-1}(\sqrt{1+x^2}+x)$.

Putting $x = \cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta + \cot\theta) = \tan^{-1}\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{1}{2}\theta = \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x.$$

$$\therefore\ \frac{dy}{dx} = -\frac{1}{2}\cdot\frac{-1}{(1+x^2)} = \frac{1}{2(1+x^2)}.$$

Hence, $\dfrac{d}{dx}\{\tan^{-1}(\sqrt{1+x^2}+x)\} = \dfrac{1}{2(1+x^2)}.$

(ii) Let $y = \tan^{-1}(\sqrt{1+x^2}-x)$.

Putting $x = \cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta - \cot\theta) = \tan^{-1}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{2(1+x^2)}.$$

Hence, $\frac{d}{dx}\{\tan^{-1}(\sqrt{1+x^2}-x)\} = \frac{-1}{2(1+x^2)}.$

EXAMPLE 19 *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)$ *w.r.t. x.*

SOLUTION Let $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right).$

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left\{\cot\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} = \left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2(1+x^2)}.$$

Hence, $\frac{d}{dx}\left\{\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)\right\} = \frac{-1}{2(1+x^2)}.$

EXAMPLE 20 *Differentiate* $\cot^{-1}(\sqrt{1+x^2}+x)$ *w.r.t. x.*

SOLUTION Let $y = \cot^{-1}(\sqrt{1+x^2}+x).$

Putting $x = \cot\theta$, we get

$$y = \cot^{-1}(\operatorname{cosec}\theta+\cot\theta) = \cot^{-1}\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$= \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \cot^{-1}\left\{\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\}$$

$$= \cot^{-1}\left(\cot\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cot^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{2(1+x^2)}.$$

Hence, $\frac{d}{dx}\{\cot^{-1}(\sqrt{1+x^2}+x)\} = \frac{-1}{2(1+x^2)}.$

EXAMPLE 21 *Differentiate* $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$ *w.r.t. x.*

SOLUTION Let $y = \cos^{-1}\left(\dfrac{x - x^{-1}}{x + x^{-1}}\right) = \cos^{-1}\left\{\dfrac{\left(x - \dfrac{1}{x}\right)}{\left(x + \dfrac{1}{x}\right)}\right\} = \cos^{-1}\left(\dfrac{x^2 - 1}{x^2 + 1}\right)$.

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) = \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}\{\cos(\pi - 2\theta)\} = (\pi - 2\theta) = \pi - 2\tan^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\pi - 2\tan^{-1}x) = \frac{d}{dx}(\pi) - 2\cdot\frac{d}{dx}(\tan^{-1}x)$$

$$= \left(0 - \frac{2}{1 + x^2}\right) = \frac{-2}{(1 + x^2)}.$$

EXAMPLE 22 *If* $y = \tan^{-1}\left(\dfrac{\sqrt{a} - \sqrt{x}}{1 + \sqrt{ax}}\right)$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $\sqrt{a} = \tan\alpha$ and $\sqrt{x} = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}\right) = \tan^{-1}\{\tan(\alpha - \theta)\} = (\alpha - \theta).$$

$$\therefore \quad y = (\alpha - \theta) \Rightarrow y = \tan^{-1}\sqrt{a} - \tan^{-1}\sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{a}) - \frac{d}{dx}(\tan^{-1}\sqrt{x})$$

$$= \left\{0 - \frac{1}{(1 + x)}\cdot\frac{1}{2}x^{-1/2}\right\} = \frac{-1}{2\sqrt{x}(1 + x)}.$$

Hence, $\dfrac{dy}{dx} = \dfrac{-1}{2\sqrt{x}(1 + x)}$.

EXAMPLE 23 *If* $y = \sin^{-1}\left\{\dfrac{5x + 12\sqrt{1 - x^2}}{13}\right\}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2004C, '05C, '08]**

SOLUTION We have

$$y = \sin^{-1}\left\{\frac{5}{13}\cdot x + \frac{12}{13}\cdot\sqrt{1 - x^2}\right\}.$$

Let $\dfrac{5}{13} = \sin\alpha$ and $x = \cos\theta$. Then,

$$\cos\alpha = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

and $\sqrt{1 - x^2} = \sqrt{1 - \cos^2\theta} = \sqrt{\sin^2\theta} = \sin\theta$.

$$\therefore \quad y = \sin^{-1}\{\sin\alpha\cos\theta + \cos\alpha\sin\theta\}$$

$$= \sin^{-1}\{\sin(\alpha + \theta)\}$$

$$= \alpha + \theta = \sin^{-1}\frac{5}{13} + \cos^{-1}x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left\{\sin^{-1}\frac{5}{13} + \cos^{-1}x\right\} = \frac{d}{dx}\left\{\sin^{-1}\frac{5}{13}\right\} + \frac{d}{dx}(\cos^{-1}x)$$

$$= \left\{0 - \frac{1}{\sqrt{1-x^2}}\right\} = \frac{-1}{\sqrt{1-x^2}}.$$

Hence, $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

EXERCISE 10D

Differentiate each of the following w.r.t. x:

1. $\sin^{-1}\left\{\sqrt{\frac{1-\cos x}{2}}\right\}$
2. $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$
3. $\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$
4. $\cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$
5. $\tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$
6. $\cot^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$
7. $\cot^{-1}\left(\sqrt{\frac{1+\cos 3x}{1-\cos 3x}}\right)$
8. $\sec^{-1}\left(\frac{1+\tan^2 x}{1-\tan^2 x}\right)$
9. $\sin^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)$
10. $\operatorname{cosec}^{-1}\left(\frac{1+\tan^2 x}{2\tan x}\right)$
11. $\cot^{-1}(\operatorname{cosec} x + \cot x)$
12. $\tan^{-1}(\cot x) + \cot^{-1}(\tan x)$
13. $\sin^{-1}\{\sqrt{1-x^2}\}$
14. $\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)$
15. $\cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$
16. $\cos^{-1}\{\sqrt{1-x^2}\}$
17. $\sin^{-1}\{2x\sqrt{1-x^2}\}$
18. $\sin^{-1}(3x - 4x^3)$
19. $\sin^{-1}(1-2x^2)$
20. $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$
21. $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
22. $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$
23. $\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$
24. $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$
25. $\sin^{-1}\left\{\frac{1}{\sqrt{1+x^2}}\right\}$
26. $\tan^{-1}\left(\frac{1+x}{1-x}\right)$
27. $\cot^{-1}\left(\frac{1+x}{1-x}\right)$
28. $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

29. $\text{cosec}^{-1}\left(\frac{1+x^2}{2x}\right)$

30. $\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

31. $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

32. $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

33. $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

34. $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}$

35. $\sin^{-1}\{2ax\sqrt{1-a^2x^2}\}$

36. $\tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\}$

37. $\sin^{-1}\left\{\frac{x^2}{\sqrt{x^4+a^4}}\right\}$

38. $\tan^{-1}\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$

39. $\cos^{-1}(2x)+2\cos^{-1}\sqrt{1-4x^2}$

40. $\tan^{-1}\left(\frac{a-x}{1+ax}\right)$

41. $\tan^{-1}\left\{\frac{\sqrt{x}-x}{1+x^{3/2}}\right\}$

42. $\tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$

43. $\tan^{-1}\left(\frac{3-2x}{1+6x}\right)$

44. $\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

45. $\tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

46. If $y=\tan^{-1}\left(\frac{ax-b}{bx+a}\right)$, prove that $\frac{dy}{dx}=\frac{1}{(1+x^2)}$.

47. If $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, show that $\frac{dy}{dx}=\frac{4}{(1+x^2)}$.

48. If $y=\sec^{-1}\left(\frac{x+1}{x-1}\right)+\sin^{-1}\left(\frac{x-1}{x+1}\right)$, show that $\frac{dy}{dx}=0$.

49. If $y=\sin\left\{2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$, show that $\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}$.

50. If $y=\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$, prove that $\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$. **[CBSE 2003]**

51. Differentiate $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ w.r.t. x.

ANSWERS (EXERCISE 10D)

1. $\frac{1}{2}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$ **4.** $\frac{1}{2}$ **5.** 1 **6.** 1 **7.** $\frac{3}{2}$ **8.** 2

9. -2 **10.** 2 **11.** $\frac{1}{2}$ **12.** -2 **13.** $\frac{-1}{\sqrt{1-x^2}}$ **14.** $\frac{-1}{2\sqrt{1-x^2}}$

15. $\frac{-1}{2\sqrt{1-x^2}}$ **16.** $\frac{1}{\sqrt{1-x^2}}$ **17.** $\frac{2}{\sqrt{1-x^2}}$ **18.** $\frac{3}{\sqrt{1-x^2}}$

19. $\dfrac{-2}{\sqrt{1-x^2}}$ **20.** $\dfrac{1}{\sqrt{1-x^2}}$ **21.** $\dfrac{1}{\sqrt{1-x^2}}$ **22.** $\dfrac{1}{2\sqrt{1-x^2}}$

23. $\dfrac{1}{\sqrt{1-x^2}}$ **24.** $\dfrac{2}{\sqrt{1-x^2}}$ **25.** $\dfrac{1}{(1+x^2)}$ **26.** $\dfrac{1}{(1+x^2)}$

27. $\dfrac{-1}{(1+x^2)}$ **28.** $\dfrac{3}{(1+x^2)}$ **29.** $\dfrac{2}{(1+x^2)}$ **30.** $\dfrac{2}{(1+x^2)}$

31. $\dfrac{-1}{(1+x^2)}$ **32.** $\dfrac{-2}{(1+x^2)}$ **33.** $\dfrac{2nx^{n-1}}{(1+x^{2n})}$ **34.** $\dfrac{1}{\sqrt{a^2-x^2}}$

35. $\dfrac{2a}{\sqrt{1-a^2x^2}}$ **36.** $\dfrac{a}{2(1+a^2x^2)}$ **37.** $\dfrac{2a^2x}{(a^4+x^4)}$ **38.** $\dfrac{-2e^{2x}}{(1+e^{4x})}$

39. $\dfrac{2}{\sqrt{1-4x^2}}$ **40.** $\dfrac{-1}{(1+x^2)}$ **41.** $\dfrac{1}{2\sqrt{x}(1+x)}-\dfrac{1}{(1+x^2)}$

42. $\dfrac{1}{2\sqrt{x}(1+x)}$ **43.** $\dfrac{-2}{(1+4x^2)}$ **44.** $\left(\dfrac{3}{1+9x^2}+\dfrac{2}{1+4x^2}\right)$

45. $\dfrac{5}{(1+25x^2)}-\dfrac{3}{(1+9x^2)}$ **51.** $\dfrac{2^{x+1}(\log 2)}{(1+4^x)}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10D)

(i) Put $x=\cos\theta$ in Q. 13, 14, 15.

(ii) Put $x=\sin\theta$ in Q. 16, 17, 18, 19, 20, 21, 22, 23, 24.

(iii) Put $x=\tan\theta$ in Q. 25, 26, 27, 28, 29, 30.

(iv) Put $x=\cot\theta$ in Q. 31, 32.

33. Put $x^n=\tan\theta$. 34. Put $x=a\sin\theta$. 35. Put $ax=\sin\theta$.

36. Put $ax=\tan\theta$. 37. Put $x^2=a^2\tan\theta$. 38. Put $e^{2x}=\cot\theta$.

39. Put $2x=\cos\theta$. 40. $y=\tan^{-1}a-\tan^{-1}x$. 41. $y=\tan^{-1}\sqrt{x}-\tan^{-1}x$.

42. $y=\tan^{-1}\sqrt{a}+\tan^{-1}\sqrt{x}$. 43. $y=\tan^{-1}3-\tan^{-1}2x$.

44. $y=\tan^{-1}3x+\tan^{-1}2x$. 45. $y=\tan^{-1}5x-\tan^{-1}3x$.

46. $y=\tan^{-1}\left[\dfrac{x-(b/a)}{(b/a)x+1}\right]=\left[\tan^{-1}x-\tan^{-1}\dfrac{b}{a}\right]$.

48. $\sec^{-1}\theta=\cos^{-1}\dfrac{1}{\theta}$ and $\cos^{-1}\theta+\sin^{-1}\theta=\dfrac{\pi}{2}$.

51. Let $y=\sin^{-1}\left\{\dfrac{2^x\cdot 2}{1+(2^x)^2}\right\}$. Puttting $2^x=\tan\theta$, we get

$$y=\sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}2^x.$$

$$\therefore\quad \frac{dy}{dx}=2\cdot\frac{d}{dx}(\tan^{-1}2^x)=2\cdot\frac{1}{\{1+(2^x)^2\}}\cdot\frac{d}{dx}(2^x)=\frac{2}{(1+4^x)}\cdot 2^x(\log 2)=\frac{2^{x+1}(\log 2)}{(1+4^x)}.$$

5. Differentiation of Implicit Function

Let $f(x, y) = a$ be a function of x and y defined in such a manner that y is not expressible directly in terms of x. Then, $f(x, y) = a$ is called an implicit function of x and y. In differentiating such a function, we differentiate both sides of the equation termwise, keeping in mind that

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}\ ;\ \frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx}\text{, and so on.}$$

SOLVED EXAMPLES

EXAMPLE 1 *If* $x^3 + y^3 = 3axy$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $x^3 + y^3 = 3axy.$ … (i)

Differentiating both sides of (i) w.r.t. x, we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a \cdot \left\{x \cdot \frac{dy}{dx} + y \cdot 1\right\}$$

$$\Rightarrow\ 3(y^2 - ax) \cdot \frac{dy}{dx} = 3(ay - x^2)$$

$$\Rightarrow\ \frac{dy}{dx} = \left(\frac{ay - x^2}{y^2 - ax}\right).$$

EXAMPLE 2 *If* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$ … (i)

Differentiating both sides of (i) w.r.t. x, we get

$$2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + 2by \cdot \frac{dy}{dx} + 2g + 2f \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow\ (2ax + 2hy + 2g) + (2hx + 2by + 2f) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow\ \frac{dy}{dx} = -\left(\frac{ax + hy + g}{hx + by + f}\right).$$

EXAMPLE 3 *If* $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, *prove that* $\dfrac{dy}{dx} = \sqrt{\dfrac{1 - y^2}{1 - x^2}}$.

SOLUTION Given: $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y).$ … (i)

Putting $x = \sin\theta$ and $y = \sin\phi$, it becomes

$$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\Rightarrow\ \frac{\cos\theta + \cos\phi}{\sin\theta - \sin\phi} = a$$

$$\Rightarrow \frac{2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)}{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)} = a$$

$$\Rightarrow \cot\left(\frac{\theta-\phi}{2}\right) = a \Rightarrow \theta-\phi = 2\cot^{-1}a$$

$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a. \quad \ldots \text{(ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0.$$

Hence, $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

EXAMPLE 4 *If* $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, *prove that* $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

SOLUTION We have $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$. … (i)

Putting $x = \sin\theta$ and $y = \cos\phi$ in (i), we get

$$\sin\theta\cos\phi + \cos\theta\sin\phi = 1$$

$$\Rightarrow \sin(\theta+\phi) = 1$$

$$\Rightarrow (\theta+\phi) = \sin^{-1}(1)$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}. \quad \ldots \text{(ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}.$$

EXAMPLE 5 *If* $x\sqrt{1+y} + y\sqrt{1+x} = 0$, *prove that* $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

SOLUTION $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow x^2(1+y) = y^2(1+x) \quad \text{[on squaring both sides]}$$

$$\Rightarrow (x-y)(x+y+xy) = 0$$

$$\Rightarrow x+y+xy = 0 \quad [\because\ x = y \text{ does not satisfy the given equation}]$$

$$\Rightarrow y = \frac{-x}{1+x}.$$

$$\therefore \quad \frac{dy}{dx} = -\left\{\frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2}\right\} = \frac{-1}{(1+x)^2} \qquad \text{[using the quotient rule].}$$

EXAMPLE 6 *If* $\sin y = x \sin (a+y)$, *prove that* $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$. **[CBSE 2012]**

SOLUTION $\sin y = x \sin (a+y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin (a+y)}. \qquad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin (a+y)\cos y - \sin y \cos (a+y)}{\sin^2(a+y)} \text{ [using the quotient rule]}$$

$$= \frac{\sin (a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}.$$

Hence, $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

EXAMPLE 7 *If* $y \cdot \sqrt{x^2+1} = \log \{\sqrt{x^2+1} - x\}$, *show that*

$$(x^2+1)\frac{dy}{dx} + xy + 1 = 0. \qquad \textbf{[CBSE 2006]}$$

SOLUTION Given: $y \cdot \sqrt{x^2+1} = \log \{\sqrt{x^2+1} - x\}$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x + (\sqrt{x^2+1}) \cdot \frac{dy}{dx}$$

$$= \frac{1}{\{\sqrt{x^2+1} - x\}} \cdot \left\{\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - 1\right\}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + (\sqrt{x^2+1})\frac{dy}{dx} = \frac{1}{\{\sqrt{x^2+1} - x\}} \cdot \left\{\frac{x}{\sqrt{x^2+1}} - 1\right\}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + (\sqrt{x^2+1})\frac{dy}{dx} = \frac{1}{\{\sqrt{x^2+1} - x\}} \cdot \frac{\{x - \sqrt{x^2+1}\}}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad xy + (x^2+1)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} + xy + 1 = 0.$$

EXAMPLE 8 *If* $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, *show that* $2x\frac{dy}{dx} + y = 2\sqrt{x}$. **[CBSE 2006]**

SOLUTION $y = \sqrt{x} + \frac{1}{\sqrt{x}} \Rightarrow \sqrt{x}\, y = x + 1$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\sqrt{x}\cdot\frac{dy}{dx}+y\cdot\frac{1}{2}x^{-1/2}=1$$

$$\Rightarrow\ \sqrt{x}\ \frac{dy}{dx}+\frac{y}{2\sqrt{x}}=1$$

$$\Rightarrow\ 2x\ \frac{dy}{dx}+y=2\sqrt{x}.$$

EXAMPLE 9 *If* $\cos(x+y)=y\sin x$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $\cos(x+y)=y\sin x.$ … (i)

On differentiating both sides of (i) w.r.t. x, we get

$$-\sin(x+y)\cdot\frac{d}{dx}(x+y)=y\cos x+\sin x\cdot\frac{dy}{dx}$$

$$\Rightarrow\ -\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right)=y\cos x+\sin x\cdot\frac{dy}{dx}$$

$$\Rightarrow\ \{\sin(x+y)+\sin x\}\cdot\frac{dy}{dx}=-\{\sin(x+y)+y\cos x\}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{-\{\sin(x+y)+y\cos x\}}{\{\sin(x+y)+\sin x\}}.$$

EXAMPLE 10 *Find* $\dfrac{dy}{dx}$ *when* $\sin(xy)+\dfrac{x}{y}=x^2-y.$

SOLUTION Given: $\sin(xy)+\dfrac{x}{y}=x^2-y.$

Differentiating both sides w.r.t. x, we get

$$\cos(xy)\cdot\frac{d}{dx}(xy)+x\cdot\left(-\frac{1}{y^2}\right)\frac{dy}{dx}+\frac{1}{y}\cdot1=2x-\frac{dy}{dx}$$

$$\Rightarrow\ \cos(xy)\cdot\left[x\cdot\frac{dy}{dx}+y\cdot1\right]-\frac{x}{y^2}\cdot\frac{dy}{dx}+\frac{1}{y}=2x-\frac{dy}{dx}$$

$$\Rightarrow\ \left[x\cos(xy)-\frac{x}{y^2}+1\right]\cdot\frac{dy}{dx}=2x-\frac{1}{y}-y\cos(xy)$$

$$\Rightarrow\ \{xy^2\cos(xy)-x+y^2\}\cdot\frac{dy}{dx}=2xy^2-y-y^3\cos(xy).$$

Hence, $\dfrac{dy}{dx}=\left\{\dfrac{2xy^2-y-y^3\cos(xy)}{xy^2\cos(xy)-x+y^2}\right\}.$

EXAMPLE 11 *If* $e^x+e^y=e^{x+y}$, *prove that* $\dfrac{dy}{dx}=-e^{y-x}.$

SOLUTION Given: $e^x+e^y=e^{x+y}.$ ... (i)

On dividing throughout by e^{x+y}, we get

$$e^{-y} + e^{-x} = 1. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$e^{-y} \cdot \left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}.$$

EXAMPLE 12 *If* $\tan^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = a$, *show that* $\frac{dy}{dx} = \frac{x(1-\tan a)}{y(1+\tan a)}$.

SOLUTION $\tan^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = a \Rightarrow \frac{(x^2-y^2)}{(x^2+y^2)} = \tan a$

$$\therefore \ (x^2-y^2) = (x^2+y^2)\tan a. \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$2x - 2y \cdot \frac{dy}{dx} = 2x \tan a + 2y \cdot \frac{dy}{dx} \cdot \tan a$$

$$\Rightarrow y(1+\tan a)\frac{dy}{dx} = x(1-\tan a)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(1-\tan a)}{y(1+\tan a)}.$$

EXERCISE 10E

Find $\frac{dy}{dx}$***, when:***

1. $x^2 + y^2 = 4$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. $\sqrt{x} + \sqrt{y} = \sqrt{a}$
4. $x^{2/3} + y^{2/3} = a^{2/3}$
5. $xy = c^2$
6. $x^2 + y^2 - 3xy = 1$
7. $xy^2 - x^2y - 5 = 0$
8. $(x^2+y^2)^2 = xy$
9. $x^2 + y^2 = \log(xy)$
10. $x^n + y^n = a^n$
11. $x \sin 2y = y \cos 2x$
12. $\sin^2 x + 2\cos y + xy = 0$
13. $y \sec x + \tan x + x^2 y = 0$
14. $\cot(xy) + xy = y$
15. $y \tan x - y^2 \cos x + 2x = 0$
16. $e^x \log y = \sin^{-1} x + \sin^{-1} y$
17. $xy \log(x+y) = 1$
18. $\tan(x+y) + \tan(x-y) = 1$
19. $\log\sqrt{x^2+y^2} = \tan^{-1}\frac{y}{x}$
20. If $y = x \sin y$, prove that $\left(x \cdot \frac{dy}{dx}\right) = \frac{y}{(1-x\cos y)}$.
21. If $xy = \tan(xy)$, show that $\frac{dy}{dx} = \frac{-y}{x}$.

22. If $y \log\ x = (x - y)$, prove that $\frac{dy}{dx} = \frac{\log\ x}{(1 + \log\ x)^2}$.

23. If $\cos y = x \cos (y + a)$, prove that $\frac{dy}{dx} = \frac{\cos^2(y + a)}{\sin\ a}$.

24. If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \tan^{-1}a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

ANSWERS (EXERCISE 10E)

1. $\frac{-x}{y}$ **2.** $\frac{-b^2x}{a^2y}$ **3.** $-\sqrt{\frac{y}{x}}$ **4.** $\frac{-y^{1/3}}{x^{1/3}}$ **5.** $\frac{-c^2}{x^2}$

6. $\frac{(2x - 3y)}{(3x - 2y)}$ **7.** $\frac{(y^2 - 2xy)}{(x^2 - 2xy)}$ **8.** $\frac{(y - 4xy^2 - 4x^3)}{(4y^3 + 4x^2y - x)}$ **9.** $\frac{y(1 - 2x^2)}{x(2y^2 - 1)}$

10. $\frac{-x^{n-1}}{y^{n-1}}$ **11.** $\frac{(2y \sin\ 2x + \sin\ 2y)}{(\cos\ 2x - 2x \cos\ 2y)}$ **12.** $\frac{(y + \sin\ 2x)}{(2 \sin\ y - x)}$

13. $\frac{-(y \sec x\ \tan\ x + \sec^2 x + 2xy)}{(x^2 + \sec\ x)}$ **14.** $\frac{-y \cot^2(xy)}{\{1 + x \cot^2(xy)\}}$

15. $\frac{(y \sec^2 x + y^2 \sin\ x + 2)}{(2y \cos\ x - \tan\ x)}$ **16.** $y \cdot \sqrt{\frac{1 - y^2}{1 - x^2}} \cdot \left\{\frac{1 - e^x \log\ y \cdot \sqrt{1 - x^2}}{(e^x \sqrt{1 - y^2}) - y}\right\}$

17. $\frac{-(x + y + x^2y)}{x^2\{y + (x + y) \log\ (x + y)\}}$ **18.** $\frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}$ **19.** $\frac{x + y}{x - y}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10E)

8. $x^4 + y^4 + 2x^2y^2 = xy$

$\Rightarrow\ 4x^3 + 4y^3 \cdot \frac{dy}{dx} + 4x^2y \cdot \frac{dy}{dx} + 4xy^2 = x\frac{dy}{dx} + y$

$\Rightarrow\ (4y^3 + 4x^2y - x)\frac{dy}{dx} = (y - 4xy^2 - 4x^3)$.

9. $x^2 + y^2 = \log\ x + \log\ y$

$\Rightarrow\ 2x + 2y\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx}$.

10. $x^n + y^n = a^n\ \Rightarrow\ nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0$.

14. $\cot\ (xy) + xy = y$

$\Rightarrow\ -\text{cosec}^2(xy) \cdot \left\{x\frac{dy}{dx} + y\right\} + \left(x\frac{dy}{dx} + y\right) = \frac{dy}{dx}$

$\Rightarrow \{\text{cosec}^2(xy) - 1\} \cdot x\frac{dy}{dx} + \frac{dy}{dx} = \{1 - \text{cosec}^2(xy)\}y$

$\Rightarrow \{x\cot^2(xy) + 1\} \cdot \frac{dy}{dx} = -y\cot^2(xy).$

17. $y\log(x+y) = \frac{1}{x}$

$\Rightarrow y \cdot \frac{1}{(x+y)} \cdot \left(1 + \frac{dy}{dx}\right) + \log(x+y) \cdot \frac{dy}{dx} = \frac{-1}{x^2}$

$\Rightarrow \left\{\frac{y}{(x+y)} + \log(x+y)\right\} \cdot \frac{dy}{dx} = -\left(\frac{1}{x^2} + \frac{y}{x+y}\right).$

19. $\frac{1}{2}\log(x^2+y^2) = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \log(x^2+y^2) = 2\tan^{-1}\frac{y}{x} \Rightarrow (x^2+y^2) = e^{2\tan^{-1}y/x}.$ ... (i)

On differentiating (i) w.r.t. x, we get

$$2x + 2y \cdot \frac{dy}{dx} = e^{2\tan^{-1}y/x} \cdot \frac{2}{\left(1 + \frac{y^2}{x^2}\right)} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2}$$

$\Rightarrow x + y\frac{dy}{dx} = (x^2+y^2) \cdot \frac{x^2}{(x^2+y^2)} \cdot \frac{\left(x\frac{dy}{dx} - y\right)}{x^2}$ [using (i)]

$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$

$\Rightarrow (x-y)\frac{dy}{dx} = (x+y) \Rightarrow \frac{dy}{dx} = \frac{(x+y)}{(x-y)}.$

24. $\frac{x^2-y^2}{x^2+y^2} = \cos\{\tan^{-1}a\} = \text{constant}$

$\Rightarrow \frac{d}{dx}\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0$

$\Rightarrow \frac{(x^2+y^2) \cdot \frac{d}{dx}(x^2-y^2) - (x^2-y^2) \cdot \frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2} = 0$

$\Rightarrow (x^2+y^2)\left[2x - 2y\frac{dy}{dx}\right] - (x^2-y^2)\left(2x + 2y\frac{dy}{dx}\right) = 0$

$\Rightarrow x\{(x^2+y^2) - (x^2-y^2)\} = y\{(x^2-y^2) + (x^2+y^2)\}\frac{dy}{dx}.$

Hence, $\frac{dy}{dx} = \frac{y}{x}.$

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6. Differentiation Using Logarithms

When the given function is a power of some expression or a product of expressions, then we take the logarithm on both sides and differentiate termwise, as shown below.

SOLVED EXAMPLES

EXAMPLE 1 *If* $y = \sqrt{\dfrac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$, *find* $\dfrac{dy}{dx}$. **[CBSE 2006]**

SOLUTION Given: $y = \sqrt{\dfrac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \frac{1}{2}\{\log (x-3) + \log (x^2+4) - \log (3x^2+4x+5)\}.$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\left\{\frac{1}{(x-3)} + \frac{2x}{(x^2+4)} - \frac{(6x+4)}{(3x^2+4x+5)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}y\right)\cdot\left\{\frac{1}{(x-3)} + \frac{2x}{(x^2+4)} - \frac{(6x+4)}{(3x^2+4x+5)}\right\}$$

$$= \frac{1}{2}\cdot\sqrt{\frac{(x-3)(x^2+4)}{(3x^2+4x+5)}}\cdot\left\{\frac{1}{(x-3)} + \frac{2x}{(x^2+4)} - \frac{(6x+4)}{(3x^2+4x+5)}\right\}.$$

EXAMPLE 2 *If* $y = \dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Given: $y = \dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log (x+4) - \frac{4}{3}\log (4x-3).$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x} + \frac{3}{2}\cdot\frac{1}{(x+4)} - \frac{4}{3}\cdot\frac{4}{(4x-3)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right]$$

$$= \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\cdot\left[\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right].$$

EXAMPLE 3 *Differentiate* $(x+1)^2(x+2)^3(x+3)^4$ *w.r.t. x.*

SOLUTION Let $y = (x+1)^2(x+2)^3(x+3)^4$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = 2\log (x+1) + 3\log (x+2) + 4\log (x+3). \quad \text{... (ii)}$$

Differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)} \right]$$

$$= (x+1)^2 (x+2)^3 (x+3)^4 \cdot \left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)} \right].$$

EXAMPLE 4 *Differentiate* $\sqrt{(x-1)(x-2)(x-3)(x-4)}$ *w.r.t. x.*

SOLUTION Let $y = \sqrt{(x-1)(x-2)(x-3)(x-4)}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \frac{1}{2}\{\log (x-1) + \log (x-2) + \log (x-3) + \log (x-4)\}. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{y}{2} \right) \cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}$$

$$= \frac{1}{2} \cdot \sqrt{(x-1)(x-2)(x-3)(x-4)}$$

$$\cdot \left\{ \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)} + \frac{1}{(x-4)} \right\}.$$

EXAMPLE 5 *Differentiate* $(e^x \cos^3 x \sin^2 x)$ *w.r.t. x.*

SOLUTION Let $y = e^x \cos^3 x \sin^2 x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x + 3 \log \cos x + 2 \log \sin x. \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{3}{\cos x} \cdot (-\sin x) + \frac{2}{\sin x} \cdot \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \{1 - 3 \tan x + 2 \cot x\}$$

$$= (e^x \cos^3 x \sin^2 x)(1 - 3 \tan x + 2 \cot x).$$

EXAMPLE 6 *Differentiate* $(\tan x \tan 2x \tan 3x \tan 4x)$ *w.r.t. x.*

SOLUTION Let $y = \tan x \tan 2x \tan 3x \tan 4x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \log \tan x + \log \tan 2x + \log \tan 3x + \log \tan 4x. \quad \text{... (ii)}$$

On diffferentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left\{ \frac{\sec^2 x}{\tan x} + \frac{2\sec^2 2x}{\tan 2x} + \frac{3\sec^2 3x}{\tan 3x} + \frac{4\sec^2 4x}{\tan 4x} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = y \cdot \left[\frac{1}{\sin x \cos x} + \frac{2}{\sin 2x \cos 2x} + \frac{3}{\sin 3x \cos 3x} + \frac{4}{\sin 4x \cos 4x} \right]$$

$$= y \cdot \left[\frac{2}{\sin 2x} + \frac{4}{\sin 4x} + \frac{6}{\sin 6x} + \frac{8}{\sin 8x} \right]$$

$$= [2 \tan x \tan 2x \tan 3x \tan 4x] \times [\operatorname{cosec} 2x + 2 \operatorname{cosec} 4x + 3 \operatorname{cosec} 6x + 4 \operatorname{cosec} 8x].$$

EXAMPLE 7 *If* $y = (2x + 3)^{(3x-5)}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = (2x + 3)^{(3x-5)}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (3x - 5) \log (2x + 3). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (3x - 5) \cdot \frac{d}{dx} \{\log (2x + 3)\} + \log (2x + 3) \cdot \frac{d}{dx} (3x - 5)$$

$$= (3x - 5) \cdot \frac{1}{(2x + 3)} \cdot 2 + \log (2x + 3) \cdot 3$$

$$\Rightarrow \quad \frac{dy}{dx} = y \cdot \left\{ \frac{(6x - 10)}{(2x + 3)} + 3 \log (2x + 3) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (2x + 3)^{(3x-5)} \cdot \left\{ \frac{(6x - 10)}{(2x + 3)} + 3 \log (2x + 3) \right\}.$$

EXAMPLE 8 *If* $y = \frac{5^x}{x^5}$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $y = \frac{5^x}{x^5}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = \log (5^x) - \log (x^5)$$

$$\Rightarrow \quad \log y = x \log 5 - 5 \log x$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = (\log 5) \cdot 1 - \frac{5}{x} \qquad \text{[differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx} = y \left(\log 5 - \frac{5}{x} \right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{5^x}{x^5} \left(\log 5 - \frac{5}{x} \right).$$

EXAMPLE 9 *Differentiate* x^x *w.r.t. x.*

SOLUTION Let $y = x^x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x \log x. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= \left(x\cdot\frac{1}{x} + \log x\cdot 1\right) = (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x(1 + \log x).$$

EXAMPLE 10 *Differentiate* $(\sin x)^x$ *w.r.t. x.*

SOLUTION Let $y = (\sin x)^x$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x \log (\sin x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{d}{dx}\{\log (\sin x)\} + \log (\sin x)\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{\sin x}\cdot\cos x + \log (\sin x)\cdot 1$$

$$= x\cot x + \log (\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\cdot[x\cot x + \log (\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x[x\cot x + \log (\sin x)].$$

EXAMPLE 11 *Differentiate* $x^{\sin^{-1}x}$ *w.r.t. x.* **[CBSE 2000]**

SOLUTION Let $y = x^{\sin^{-1}x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\sin^{-1}x)(\log x). \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\sin^{-1}x)\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(\sin^{-1}x)$$

$$= (\sin^{-1}x)\,\frac{1}{x} + (\log x)\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left[\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1}x}\cdot\left\{\frac{\sin^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}.$$

EXAMPLE 12 *Differentiate* $(\sin x)^{\log x}$ *w.r.t. x.*

SOLUTION Let $y = (\sin x)^{\log x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = (\log x)(\log \sin x). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = (\log x)\cdot\frac{d}{dx}(\log \sin x) + (\log \sin x)\cdot\frac{d}{dx}(\log x)$$

$$= (\log x)\cdot\frac{1}{\sin x}\cdot\cos x + (\log \sin x)\cdot\frac{1}{x}$$

$$= (\log x)\cot x + \frac{(\log \sin x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left[(\log x)\cot x + \frac{(\log \sin x)}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\cdot\left[(\log x)\cot x + \frac{(\log \sin x)}{x}\right].$$

EXAMPLE 13 *Differentiate* $x^x \sin^{-1}\sqrt{x}$ *w.r.t. x.* **[CBSE 2005C]**

SOLUTION Let $y = x^x \sin^{-1}\sqrt{x}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x\log x + \log(\sin^{-1}\sqrt{x}). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(x\log x) + \frac{d}{dx}\{\log(\sin^{-1}\sqrt{x})\}$$

$$= \left(x\cdot\frac{1}{x} + \log x\cdot 1\right) + \frac{1}{\sin^{-1}\sqrt{x}}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2}x^{-1/2}$$

$$= \left\{(1+\log x) + \frac{1}{2(\sqrt{x-x^2})\sin^{-1}\sqrt{x}}\right\}$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left\{(1+\log x) + \frac{1}{2(\sqrt{x-x^2})\sin^{-1}\sqrt{x}}\right\}$$

$$= (x^x \sin^{-1}\sqrt{x})\left\{(1+\log x) + \frac{1}{2(\sqrt{x-x^2})\sin^{-1}\sqrt{x}}\right\}.$$

$$= \left\{(x^x \sin^{-1}\sqrt{x})(1+\log x) + \frac{x^x}{2(\sqrt{x-x^2})}\right\}.$$

EXAMPLE 14 *Differentiate* $\dfrac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$ *w.r.t. x.*

SOLUTION Let $y = \dfrac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$. ... (i)

Taking logarithm on both sides of (i), we get

$$\log y = x^2 + \log(\tan^{-1}x) - \frac{1}{2}\log(1+x^2). \quad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2x + \frac{1}{\tan^{-1}x}\cdot\frac{1}{(1+x^2)} - \frac{1}{2}\cdot\frac{2x}{(1+x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[2x + \frac{1}{(1+x^2)\tan^{-1}x} - \frac{x}{(1+x^2)}\right]$$

$$= \frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}\cdot\left[2x + \frac{1}{(1+x^2)\tan^{-1}x} - \frac{x}{(1+x^2)}\right].$$

EXAMPLE 15 *Differentiate* $\left\{x^{\tan x} + \sqrt{\frac{x^2+1}{x}}\right\}$ *w.r.t. x.* **[CBSE 2006C]**

SOLUTION Let $y = u + v$, where $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2+1}{x}}$.

Now, $u = x^{\tan x}$

$$\Rightarrow \quad \log u = (\tan x)(\log x)$$

$$\Rightarrow \quad \frac{1}{u}\cdot\frac{du}{dx} = (\tan x)\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(\tan x)$$

[differentiating w.r.t. x]

$$= (\tan x)\cdot\frac{1}{x} + (\log x)\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\cdot\left[\frac{\tan x}{x} + (\log x)\sec^2 x\right]$$

$$\Rightarrow \quad \frac{du}{dx} = x^{\tan x}\cdot\left\{\frac{\tan x}{x} + (\log x)\sec^2 x\right\}. \quad \text{... (i)}$$

And, $v = \sqrt{\frac{x^2+1}{x}}$

$$\Rightarrow \quad \log v = \frac{1}{2}\cdot\{\log(x^2+1) - \log x\}$$

$$\Rightarrow \quad \frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{2}\cdot\left\{\frac{2x}{(x^2+1)} - \frac{1}{x}\right\}$$ [differentiating w.r.t. x]

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v}{2}\cdot\left\{\frac{2x^2-(x^2+1)}{x(x^2+1)}\right\}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{2}\sqrt{\frac{x^2+1}{x}}\cdot\left\{\frac{(x^2-1)}{x(x^2+1)}\right\}. \quad \text{... (ii)}$$

$$\therefore \quad y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{\tan x} \cdot \left\{ \frac{\tan x}{x} + (\log x)\sec^2 x \right\} + \frac{1}{2} \cdot \sqrt{\frac{x^2+1}{x}} \cdot \left\{ \frac{(x^2-1)}{x(x^2+1)} \right\}.$$

EXAMPLE 16 *If* $y = (x)^{\cos x} + (\cos x)^{\sin x}$, *find* $\frac{dy}{dx}$. **[CBSE 2005C, '06, '09]**

SOLUTION Let $y = u + v$, where $u = (x)^{\cos x}$ and $v = (\cos x)^{\sin x}$.

Now, $u = (x)^{\cos x}$

$$\Rightarrow \log u = (\cos x)(\log x)$$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (\cos x) \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(\cos x)$$

[on differentiating w.r.t. x]

$$= (\cos x) \cdot \frac{1}{x} + (\log x)(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = u \cdot \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}$$

$$\Rightarrow \frac{du}{dx} = (x)^{\cos x} \left\{ \frac{\cos x}{x} - (\log x)(\sin x) \right\}. \quad \text{... (i)}$$

And, $v = (\cos x)^{\sin x}$

$$\Rightarrow \log v = (\sin x) \log (\cos x)$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) \cdot \frac{d}{dx}\{\log (\cos x)\} + \log (\cos x) \cdot \frac{d}{dx}(\sin x)$$

[on differentiating w.r.t. x]

$$\Rightarrow \frac{dv}{dx} = v \cdot \left\{ (\sin x) \cdot \frac{(-\sin x)}{\cos x} + \log (\cos x) \cdot \cos x \right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}. \quad \text{... (ii)}$$

$$\therefore \quad y = (u + v)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x)^{\cos x} \cdot \left\{ \frac{\cos x}{x} - (\log x) \sin x \right\}$$

$$+ (\cos x)^{\sin x} \cdot \{-\sin x \tan x + \cos x \cdot \log (\cos x)\}.$$

EXAMPLE 17 *If* $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, *find* $\frac{dy}{dx}$.

SOLUTION Let $y = u + v$, where $u = (\sin x)^{\tan x}$ and $v = (\cos x)^{\sec x}$.

Now, $u = (\sin x)^{\tan x}$

$$\Rightarrow \log u = (\tan x)(\log \sin x)$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = (\tan x)\cdot\frac{d}{dx}(\log \sin x) + (\log \sin x)\cdot\frac{d}{dx}(\tan x)$$

[on differentiating w.r.t. x]

$$= (\tan x)\cdot\frac{1}{\sin x}\cdot\cos x + (\log \sin x)\cdot\sec^2 x$$

$$\Rightarrow \frac{du}{dx} = u[1 + (\log \sin x)\sec^2 x]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\tan x}\cdot\{1 + (\log \sin x)\sec^2 x\}. \quad \text{... (i)}$$

And, $v = (\cos x)^{\sec x}$

$$\Rightarrow \log v = (\sec x)\cdot\log(\cos x)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\sec x)\cdot\frac{d}{dx}\{\log(\cos x)\} + \log(\cos x)\cdot\frac{d}{dx}(\sec x)$$

[on differentiating w.r.t. x]

$$= (\sec x)\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log(\cos x)\cdot\sec x\tan x$$

$$= (\sec x\tan x)[\log(\cos x) - 1]$$

$$\Rightarrow \frac{dv}{dx} = v\cdot(\sec x\tan x)[\log(\cos x) - 1]$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sec x}\cdot(\sec x\tan x)[\log(\cos x) - 1]. \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}\cdot\{1 + (\log \sin x)\sec^2 x\}$$

$$+ (\cos x)^{\sec x}\cdot(\sec x\tan x)[\log(\cos x) - 1] \text{ [from (i) and (ii)].}$$

EXAMPLE 18 *If* $x^y = y^x$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $x^y = y^x$

$$\Rightarrow y\log x = x\log y. \quad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$y\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(y) = x\cdot\frac{d}{dx}(\log y) + (\log y)\cdot\frac{d}{dx}(x)$$

$$\Rightarrow y\cdot\frac{1}{x} + (\log x)\cdot\frac{dy}{dx} = x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + (\log y)\cdot 1$$

$$\Rightarrow \left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \left(\log y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y\log x - x)}{y}\cdot\frac{dy}{dx} = \frac{(x\log y - y)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

EXAMPLE 19 *If* $x^y \cdot y^x = 1$, *find* $\frac{dy}{dx}$.

SOLUTION Given: $x^y \cdot y^x = 1$

$\Rightarrow (y \log x) + (x \log y) = 0.$ ... (i) $[\because \log 1 = 0]$

On differentiating both sides of (i) w.r.t. x, we get

$$y \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(y) + x \cdot \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x) = 0$$

$$\Rightarrow y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1 = 0$$

$$\Rightarrow \left(\log x + \frac{x}{y}\right) \cdot \frac{dy}{dx} = -\left(\log y + \frac{y}{x}\right)$$

$$\Rightarrow \frac{(y \log x + x)}{y} \cdot \frac{dy}{dx} = \frac{-(x \log y + y)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y(x \log y + y)}{x(y \log x + x)}.$$

EXAMPLE 20 *If* $x^y = e^{x-y}$, *prove that* $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$. **[CBSE 2010C]**

SOLUTION We have

$$x^y = e^{x-y} \Rightarrow y \log x = (x - y)$$

$$\Rightarrow (1 + \log x)y = x$$

$$\Rightarrow y = \frac{x}{(1 + \log x)}.$$ ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{(1 + \log x - 1)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}.$$

EXAMPLE 21 *If* $x^y + y^x = a^b$, *find* $\frac{dy}{dx}$.

SOLUTION Let $u = x^y$ and $v = y^x$.

Then, $u + v = a^b \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0.$ ... (i) $[\because a^b = \text{constant}]$

Now, $u = x^y \Rightarrow \log u = y \log x$ [taking log on both sides]

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$ [on differentiation]

$$\Rightarrow \frac{du}{dx} = u\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left[\frac{y + x\log x \cdot \frac{dy}{dx}}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{y-1}\left[y + x\log x \cdot \frac{dy}{dx}\right].$$

And, $v = y^x \Rightarrow \log v = x\log y$ [taking log on both sides]

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \log y \quad \text{[on differentiation]}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\left[\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right] \Rightarrow \frac{dv}{dx} = y^x\left\{\frac{x\cdot\frac{dy}{dx} + y\log y}{y}\right\}$$

$$\Rightarrow \frac{dv}{dx} = y^{(x-1)}\cdot\left\{x\cdot\frac{dy}{dx} + y\log y\right\}.$$

Using (i), we get $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$\Rightarrow x^{(y-1)}\left\{y + x\log x\cdot\frac{dy}{dx}\right\} + y^{(x-1)}\cdot\left\{x\frac{dy}{dx} + y\log y\right\} = 0$$

$$\Rightarrow \{x^y(\log x) + x\cdot y^{(x-1)}\}\cdot\frac{dy}{dx} = -\{y\cdot x^{(y-1)} + y^x(\log y)\}.$$

$$\therefore \frac{dy}{dx} = \frac{-\{y\cdot x^{(y-1)} + y^x(\log y)\}}{\{x^y(\log x) + x\,y^{(x-1)}\}}.$$

EXAMPLE 22 *If* $x^x + x^y + y^x = a^b$, *find* $\frac{dy}{dx}$.

SOLUTION Let $u = x^x$, $v = x^y$ and $w = y^x$. Then,

$$u + v + w = a^b$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0. \quad \text{... (i)} \quad [\because\ a^b = \text{constant}]$$

Now, $u = x^x$

$$\Rightarrow \log u = x\log x$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

[on differentiating both sides w.r.t. x]

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{x\cdot\frac{1}{x} + (\log x)\cdot 1\right\}$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x). \quad \text{... (ii)}$$

And, $v = x^y$

$\Rightarrow \log v = y \log x$

$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = y \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(y)$

[on differentiating both sides w.r.t. x]

$\Rightarrow \frac{dv}{dx} = v \cdot \left[y \cdot \frac{1}{x} + (\log x) \cdot \frac{dy}{dx} \right]$

$\Rightarrow \frac{dv}{dx} = x^y \left\{ \frac{y}{x} + (\log x) \frac{dy}{dx} \right\}$. ... (iii)

And, $w = y^x$

$\Rightarrow \log w = x \log y$

$\Rightarrow \frac{1}{w} \cdot \frac{dw}{dx} = x \cdot \frac{d}{dx}(\log y) + (\log y) \cdot \frac{d}{dx}(x)$

[on differentiating both sides w.r.t. x]

$\Rightarrow \frac{dw}{dx} = w \cdot \left\{ x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + (\log y) \cdot 1 \right\}$

$\Rightarrow \frac{dw}{dx} = y^x \cdot \left\{ \frac{x}{y} \cdot \frac{dy}{dx} + (\log y) \right\}$. ... (iv)

Using (ii), (iii) and (iv) in (i), we get

$$x^x(1 + \log x) + x^y \left\{ \frac{y}{x} + (\log x) \frac{dy}{dx} \right\} + y^x \cdot \left\{ \frac{x}{y} \cdot \frac{dy}{dx} + (\log y) \right\} = 0$$

$\Rightarrow \{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\} + \{x^y(\log x) + xy^{(x-1)}\} \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-\{x^x(1 + \log x) + y \cdot x^{(y-1)} + y^x(\log y)\}}{\{x^y(\log x) + xy^{(x-1)}\}}$.

EXAMPLE 23 *If* $y = x^{(x^x)}$, *find* $\frac{dy}{dx}$.

SOLUTION Let $x^x = u$. Then, $y = x^u$.

$\therefore \; x \log x = \log u$ and $\log y = u \log x$.

Now, $\log u = x \log x$

$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$ [on differentiating w.r.t. x]

$\Rightarrow \frac{du}{dx} = u \cdot \left[x \cdot \frac{1}{x} + \log x \cdot 1 \right]$

$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$. ... (i)

And, $\log y = u \log x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = u \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(u)$

$= u \cdot \frac{1}{x} + (\log x) \cdot \frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left[\frac{u}{x} + (\log x) \cdot \frac{du}{dx}\right]$$

$$= x^{(x^x)} \cdot \left[\frac{x^x}{x} + (\log x)\{x^x(1 + \log x)\}\right]$$

$$= x^{(x^x)} \cdot [x^{(x-1)} + x^x(\log x) + x^x(\log x)^2].$$

EXAMPLE 24 *If* $y = (\log x)^{\cos x} + \frac{x^2+1}{x^2-1}$, *find* $\frac{dy}{dx}$. **[CBSE 2008C]**

SOLUTION Let $(\log x)^{\cos x} = u$ and $\frac{x^2+1}{x^2-1} = v$. Then,

$$u = (\log x)^{\cos x} \Rightarrow \log u = \cos x \cdot \log (\log x). \quad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{d}{dx}\{\log (\log x)\} + \log (\log x) \cdot \frac{d}{dx}(\cos x)$$

$$= \cos x \frac{1}{\log x} \cdot \frac{1}{x} - (\sin x) \cdot \log (\log x).$$

$$\therefore \frac{du}{dx} = u \cdot \left\{\frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x)\right\}$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{\cos x} \cdot \left\{\frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x)\right\}.$$

$$\text{And, } v = \frac{(x^2+1)}{(x^2-1)} \Rightarrow \frac{dv}{dx} = \frac{(x^2-1) \cdot \frac{d}{dx}(x^2+1) - (x^2+1) \cdot \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\log x)^{\cos x} \cdot \left\{\frac{\cos x}{x \log x} - (\sin x) \cdot \log (\log x)\right\} - \frac{4x}{(x^2-1)^2}.$$

EXAMPLE 25 *Differentiate* $(\log x)^x + x^{\log x}$ *w.r.t. x.* **[CBSE 2013]**

SOLUTION Let $y = (\log x)^x + x^{\log x}$.

Let $(\log x)^x = u$ and $x^{\log x} = v$. Then,

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \quad \text{... (i)}$$

$$\text{Now, } u = (\log x)^x \Rightarrow \log u = x \log (\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log (\log x) \cdot 1$$

$$\Rightarrow \quad \frac{du}{dx} = u \cdot \left\{ \frac{1}{\log x} + \log (\log x) \right\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x \cdot \left\{ \frac{1}{\log x} + \log (\log x) \right\}. \qquad \ldots \text{(ii)}$$

$$v = x^{\log x} \Rightarrow \log v = (\log x)(\log x) = (\log x)^2$$

$$\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} = 2\log x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{2\log x}{x} \right)$$

$$\Rightarrow \frac{dv}{dx} = x^{\log x} \cdot \frac{2\log x}{x} = x^{\log x - 1} \cdot 2\log x. \qquad \ldots \text{(iii)}$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i), we get

$$\frac{dy}{dx} = (\log x)^{x-1} + (\log x)^x \times \log (\log x) + x^{\log x - 1} \cdot 2\log x.$$

EXAMPLE 26 *If* $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, *find* $\frac{dy}{dx}$. **[CBSE 2013C]**

SOLUTION Let $(\sin x)^x = u$ and $\sin^{-1}\sqrt{x} = v$. Then,

$$y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \qquad \ldots \text{(i)}$$

Now, $u = (\sin x)^x \Rightarrow \log u = x \log (\sin x)$

$$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log (\sin x) \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\{x \cot x + \log (\sin x)\}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \cdot \{x \cot x + \log (\sin x)\}. \qquad \ldots \text{(ii)}$$

$$v = \sin^{-1}\sqrt{x} \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{(1-x)}} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{(1-x)}} = \frac{1}{2\sqrt{x - x^2}}.$$

$\ldots$ (iii)

Using (ii) and (iii) in (i), we get

$$\frac{dy}{dx} = (\sin x)^x \{x \cot x + \log (\sin x)\} + \frac{1}{2\sqrt{x - x^2}}.$$

EXAMPLE 27 *If* $(\tan^{-1}x)^y + y^{\cot x} = 1$, *then find* $\frac{dy}{dx}$. **[CBSE 2014C]**

SOLUTION Let $(\tan^{-1}x)^y = u$ and $y^{\cot x} = v$. Then,

$$u + v = 1 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0. \qquad \ldots \text{(i)}$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow \log u = y \log (\tan^{-1}x)$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{(1+x^2)} + \log(\tan^{-1}x)\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{\frac{y}{(1+x^2)\tan^{-1}x} + \log(\tan^{-1}x)\cdot\frac{dy}{dx}\right\}$$

$$\Rightarrow \frac{du}{dx} = (\tan^{-1}x)^y\cdot\left\{\frac{y}{(1+x^2)\tan^{-1}x} + \log(\tan^{-1}x)\cdot\frac{dy}{dx}\right\}$$

$$\Rightarrow \frac{du}{dx} = \frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)} + (\tan^{-1}x)^y\log(\tan^{-1}x)\cdot\frac{dy}{dx}. \qquad \ldots \text{(ii)}$$

Again, $v = y^{\cot x}$

$$\Rightarrow \log v = (\cot x)\log y$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\cot x)\cdot\frac{1}{y}\cdot\frac{dy}{dx} + (\log y)(-\operatorname{cosec}^2 x)$$

$$\Rightarrow \frac{dv}{dx} = v\left\{\frac{\cot x}{y}\cdot\frac{dy}{dx} - (\operatorname{cosec}^2 x)(\log y)\right\}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left\{\frac{\cot x}{y}\cdot\frac{dy}{dx} - (\operatorname{cosec}^2 x)(\log y)\right\}$$

$$\Rightarrow \frac{dv}{dx} = (\cot x)y^{(\cot x)-1}\cdot\frac{dy}{dx} - y^{\cot x}(\operatorname{cosec}^2 x)(\log y). \qquad \ldots \text{(iii)}$$

Using (ii) and (iii) in (i), we get

$$\frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)} + (\tan^{-1}x)^y\log(\tan^{-1}x)\frac{dy}{dx} + (\cot x)y^{(\cot x)-1}\cdot\frac{dy}{dx} - y^{\cot x}(\operatorname{cosec}^2 x)(\log y) = 0$$

$$\Rightarrow \{(\tan^{-1}x)^y\log(\tan^{-1}x) + (\cot x)y^{(\cot x)-1}\}\frac{dy}{dx} = \left\{y^{\cot x}(\operatorname{cosec}^2 x)(\log y) - \frac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^{\cot x}(\operatorname{cosec}^2 x)(\log y) - \dfrac{y(\tan^{-1}x)^{y-1}}{(1+x^2)}}{(\tan^{-1}x)^y\log(\tan^{-1}x) + (\cot x)y^{(\cot x)-1}}.$$

EXERCISE 10F

Find $\frac{dy}{dx}$, when:

1. $y = x^{1/x}$
2. $y = x^{\sqrt{x}}$
3. $y = (\log x)^x$
4. $y = x^{\sin x}$
5. $y = x^{(\cos^{-1}x)}$
6. $y = (\tan x)^{1/x}$

7. $y = (\sin x)^{\cos x}$
8. $y = (\log x)^{\sin x}$
9. $y = (\cos x)^{\log x}$
10. $y = (\tan x)^{\sin x}$
11. $y = (\cos x)^{\cos x}$
12. $y = (\tan x)^{\cot x}$
13. $y = x^{\sin 2x}$
14. $y = (\sin^{-1} x)^x$
15. $y = \sin(x^x)$
16. $y = (3x+5)^{(2x-3)}$
17. $y = (x+1)^3 (x+2)^4 (x+3)^5$
18. $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
19. $y = (2-x)^3 (3+2x)^5$
20. $y = \cos x \cos 2x \cos 3x$
21. $y = \dfrac{x^5\sqrt{x+4}}{(2x+3)^2}$
22. $y = \dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3 \cdot e^x}$
23. $y = \dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}$
24. $y = \dfrac{x^2\sqrt{1+x}}{(1+x^2)^{3/2}}$
25. $y = \sqrt{(x-2)(2x-3)(3x-4)}$
26. $y = \sin 2x \sin 3x \sin 4x$
27. $y = \dfrac{x^3 \sin x}{e^x}$
28. $y = \dfrac{e^x \log x}{x^2}$
29. $y = \dfrac{x\cos^{-1} x}{\sqrt{1-x^2}}$
30. $y = (1+x)(1+x^2)(1+x^4)(1+x^6)$
31. $y = x^x - 2^{\sin x}$
32. $y = (\log x)^x + x^{\log x}$
33. $y = x^{\sin x} + (\sin x)^{\cos x}$
34. $y = (x\cos x)^x + (x \sin x)^{1/x}$
35. $y = (\sin x)^x + \sin^{-1}\sqrt{x}$
36. $y = x^{x\cos x} + \left(\dfrac{x^2+1}{x^2-1}\right)$ **[CBSE 2011]**
37. $y = e^x \sin^3 x \cos^4 x$
38. $y = 2^x \cdot e^{3x} \sin 4x$
39. $y = x^x \cdot e^{(2x+5)}$
40. $y = (2x+3)^5 (3x-5)^7 (5x-1)^3$
41. $(\cos x)^y = (\cos y)^x$ **[CBSE 2012]**
42. $(\tan x)^y = (\tan y)^x$
43. $y = (\log x)^x + (x)^{\log x}$ **[CBSE 2004C]**
44. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\dfrac{dy}{dx} = (xy+1)$.
45. If $y = \sqrt{x+y}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{(2y-1)}$.
46. If $x^a y^b = (x+y)^{(a+b)}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$.
47. If $(x^x + y^x) = 1$, show that $\dfrac{dy}{dx} = -\left\{\dfrac{x^x(1+\log x) + y^x(\log y)}{xy^{x-1}}\right\}$
48. If $y = e^{\sin x} + (\tan x)^x$, prove that
$$\frac{dy}{dx} = e^{\sin x}\cos x + (\tan x)^x [2x \operatorname{cosec} 2x + \log \tan x].$$ **[CBSE 2003]**
49. If $y = \log(x + \sqrt{1+x^2})$, prove that
$$\frac{dy}{dx} = \frac{1}{\log(x+\sqrt{1+x^2})} \cdot \frac{1}{\sqrt{1+x^2}}.$$ **[CBSE 2003]**

50. If $y = \log \sin \sqrt{x^2+1}$, prove that $\frac{dy}{dx} = \frac{x \cot \sqrt{x^2+1}}{\sqrt{x^2+1}}$. **[CBSE 2003]**

51. If $y = \log \sqrt{\frac{1-\cos x}{1+\cos x}}$, show that $\frac{dy}{dx} = \text{cosec } x$. **[CBSE 2003]**

52. If $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, show that $\frac{dy}{dx} = \sec x$. **[CBSE 2002]**

53. If $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$, show that $\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$. **[CBSE 2002]**

54. If $y = \log \sqrt{\frac{1+\cos^2 x}{1-e^{2x}}}$, show that $\frac{dy}{dx} = \frac{e^{2x}}{(1-e^{2x})} - \frac{\sin x \cos x}{(1+\cos^2 x)}$. **[CBSE 2003C]**

55. If $y = (x)^{\cos x} + (\sin x)^{\tan x}$, prove that
$\frac{dy}{dx} = x^{\cos x}\left\{\frac{\cos x}{x} - (\sin x) \log x\right\} + (\sin x)^{\tan x} \cdot \{1 + (\log \sin x) \sec^2 x\}$.
[CBSE 2004C, '09]

56. If $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$, prove that
$\frac{dy}{dx} = (\sin x)^{\cos x} \cdot [\cot x \cos x - \sin x (\log \sin x)]$
$+ (\cos x)^{\sin x} \cdot [\cos x (\log \cos x) - \sin x \tan x]$.

57. If $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$, prove that
$\frac{dy}{dx} = (\tan x)^{\cot x} . \text{cosec}^2 x (1 - \log \tan x) + (\cot x)^{\tan x} \cdot \sec^2 x [\log (\cot x) - 1]$.

58. If $y = x^{\cos x} + (\cos x)^x$, prove that
$\frac{dy}{dx} = x^{\cos x} \cdot \left\{\frac{\cos x}{x} - (\sin x) \log x\right\} + (\cos x)^x [(\log \cos x) - x \tan x]$.

59. If $y = x^{\log x} + (\log x)^x$, prove that
$\frac{dy}{dx} = x^{(\log x)}\left\{\frac{2 \log x}{x}\right\} + (\log x)^x \cdot \left\{\frac{1}{\log x} + \log (\log x)\right\}$.

60. If $y = x^{(x^2-3)} + (x-3)^{x^2}$, find $\frac{dy}{dx}$.

61. If $f(x) = \left(\frac{3+x}{1+x}\right)^{2+3x}$, find $f'(0)$. **[CBSE 2005C]**

62. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. **[CBSE 2009]**

63. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$. **[CBSE 2009]**

64. If $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$, find $\frac{dy}{dx}$. **[CBSE 2012C]**

65. If $y = \tan^{-1}\frac{a}{x} + \log \sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{(x^4 - a^4)}$. **[CBSE 2014C]**

66. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$. **[CBSE 2014]**

ANSWERS (EXERCISE 10F)

1. $\dfrac{x^{1/x}(1-\log x)}{x^2}$ **2.** $\dfrac{x^{\sqrt{x}}(2+\log x)}{2\sqrt{x}}$ **3.** $(\log x)^x\left\{\dfrac{1}{\log x}+\log(\log x)\right\}$

4. $x^{\sin x}\left[\dfrac{\sin x}{x}+(\cos x)\log x\right]$ **5.** $x^{\cos^{-1}x}\left[\dfrac{\cos^{-1}x}{x}-\dfrac{\log x}{\sqrt{1-x^2}}\right]$

6. $(\tan x)^{1/x}\left\{\dfrac{2x\operatorname{cosec}2x-\log(\tan x)}{x^2}\right\}$

7. $(\sin x)^{\cos x}\cdot\{\cot x\cos x-\sin x\log(\sin x)\}$

8. $(\log x)^{\sin x}\cdot\left\{\dfrac{\sin x}{x\log x}+\cos x\cdot\log(\log x)\right\}$

9. $(\cos x)^{\log x}\cdot\left\{\dfrac{\log(\cos x)}{x}-(\tan x)(\log x)\right\}$

10. $(\tan x)^{\sin x}\cdot\{\cos x\cdot\log(\tan x)+\sec x\}$

11. $(\cos x)^{\cos x}\cdot\{(-\sin x)(1+\log\cos x)\}$

12. $(\tan x)^{\cot x}\cdot\{\operatorname{cosec}^2x(1-\log\tan x)\}$

13. $x^{\sin 2x}\cdot\left\{\dfrac{\sin 2x}{x}+(2\cos 2x)\log x\right\}$

14. $(\sin^{-1}x)^x\cdot\left\{\log(\sin^{-1}x)+\dfrac{x}{(\sin^{-1}x)\sqrt{1-x^2}}\right\}$ **15.** $[\cos(x^x)]\,x^x(1+\log x)$

16. $(3x+5)^{(2x-3)}\cdot\left\{\dfrac{3(2x-3)}{(3x+5)}+2\log(3x+5)\right\}$

17. $(x+1)^3(x+2)^4(x+3)^5\cdot\left\{\dfrac{3}{(x+1)}+\dfrac{4}{(x+2)}+\dfrac{5}{(x+3)}\right\}$

18. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\cdot\left[\dfrac{1}{(x-1)}+\dfrac{1}{(x-2)}-\dfrac{1}{x-3}-\dfrac{1}{(x-4)}-\dfrac{1}{(x-5)}\right]$

19. $(2-x)^3(3+2x)^5\cdot\left[\dfrac{10}{(3+2x)}-\dfrac{3}{(2-x)}\right]$

20. $-(\cos x\cos 2x\cos 3x)\cdot[\tan x+2\tan 2x+3\tan 3x]$

21. $\dfrac{x^5\sqrt{x+4}}{(2x+3)^2}\cdot\left\{\dfrac{5}{x}+\dfrac{1}{2(x+4)}-\dfrac{4}{(2x+3)}\right\}$

22. $\dfrac{(x+1)^2\sqrt{x-1}}{(x+4)^3\cdot e^x}\cdot\left\{\dfrac{2}{(x+1)}+\dfrac{1}{2(x-1)}-\dfrac{3}{(x+4)}-1\right\}$

23. $\dfrac{\sqrt{x}(3x+5)^2}{\sqrt{x+1}}\cdot\left\{\dfrac{1}{2x}+\dfrac{6}{(3x+5)}-\dfrac{1}{2(x+1)}\right\}$

24. $\dfrac{x^2\sqrt{1+x}}{(1+x^2)^{3/2}}\cdot\left\{\dfrac{2}{x}+\dfrac{1}{2(1+x)}-\dfrac{3x}{(1+x^2)}\right\}$

25. $\dfrac{1}{2}\sqrt{(x-2)(2x-3)(3x-4)}\cdot\left\{\dfrac{1}{(x-2)}+\dfrac{2}{(2x-3)}+\dfrac{3}{(3x-4)}\right\}$

26. $(\sin 2x \sin 3x \sin 4x)(2\cot 2x + 3\cot 3x + 4\cot 4x)$

27. $\dfrac{x^3 \sin x}{e^x} \cdot \left(\dfrac{3}{x} + \cot x - 1\right)$ **28.** $\dfrac{e^x \log x}{x^2} \cdot \left\{1 + \dfrac{1}{(x \log x)} - \dfrac{2}{x}\right\}$

29. $\dfrac{x \cos^{-1} x}{\sqrt{1-x^2}} \cdot \left\{\dfrac{1}{x} - \dfrac{1}{(\cos^{-1} x)\sqrt{1-x^2}} + \dfrac{x}{(1-x^2)}\right\}$

30. $(1+x)(1+x^2)(1+x^4)(1+x^6) \cdot \left\{\dfrac{1}{(1+x)} + \dfrac{2x}{(1+x^2)} + \dfrac{4x^3}{(1+x^4)} + \dfrac{6x^5}{(1+x^6)}\right\}$

31. $x^x(1 + \log x) + (\log 2) \cdot 2^{\sin x}(\cos x)$

32. $(\log x)^x \cdot \left\{\dfrac{1}{\log x} + \log(\log x)\right\} + e^{(\log x)^2} \cdot \dfrac{2 \log x}{x}$

33. $x^{\sin x} \cdot \left\{\dfrac{\sin x}{x} + \cos x \cdot \log x\right\} + (\sin x)^{\cos x} \cdot \{\cos x \cot x - \sin x \log(\sin x)\}$

34. $(x \cos x)^x \{1 - x \tan x + \log x + \log \cos x\}$

$$+ (x \sin x)^{1/x} \cdot \left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log \sin x}{x^2}\right\}$$

35. $(\sin x)^x \cdot \{x \cot x + \log \sin x\} + \dfrac{1}{2\sqrt{x - x^2}}$

36. $x^{x \cos x} \cdot \{\cos x - x \sin x (\log x) + \cos x (\log x)\} - \dfrac{4x}{(x^2-1)^2}$

37. $e^x \sin^3 x \cos^4 x (1 + 3\cot x - 4\tan x)$

38. $2^x \cdot e^{3x} \sin 4x \cdot \{(\log 2) + 3 + 4\cot 4x\}$ **39.** $x^x \cdot e^{(2x+5)} \cdot (3 + \log x)$

40. $(2x+3)^5 (3x-5)^7 (5x-1)^3 \cdot \left[\dfrac{10}{(2x+3)} + \dfrac{21}{(3x-5)} + \dfrac{15}{(5x-1)}\right]$

41. $\dfrac{(\log \cos y + y \tan x)}{(\log \cos x + x \tan y)}$ **42.** $\dfrac{\log \tan y - 2y \operatorname{cosec} 2x}{\log \tan x - 2x \operatorname{cosec} 2y}$

43. $(\log x)^x \left\{\log(\log x) + \dfrac{1}{\log x}\right\} + (x)^{\log x} \cdot \dfrac{2}{x} \log x$

60. $x^{(x^2-3)} \cdot \left\{\dfrac{(x^2-3)}{x} + (2x) \log x\right\} + (x-3)^{x^2} \cdot \left\{\dfrac{x^2}{(x-3)} + 2x \log(x-3)\right\}$

61. $(27 \log 3) - 12$ **62.** $(\sin x)^x [x \cot x + \log(\sin x)] + \dfrac{1}{2\sqrt{x - x^2}}$

63. $\dfrac{y(3x^2 - y^2)}{x(x^2 - 3y^2)}$ **64.** $x^{\cot x}\left(\dfrac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) + \dfrac{2x^2 + 14x + 3}{(x^2 + x + 2)^2}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10F)

15. Let $x^x = t$. Then,

$$\log t = x \log x \Rightarrow \frac{1}{t} \cdot \frac{dt}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \frac{dt}{dx} = t[1 + \log x] = x^x(1 + \log x).$$

$\therefore\ y = \sin t \Rightarrow \dfrac{dy}{dt} = \cos t = \cos x^x.$

$\therefore\ \dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)} = [\cos(x^x)]\, x^x(1 + \log x).$

18. $\log y = \dfrac{1}{2} \cdot \{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\}$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{1}{2} \cdot \left\{\dfrac{1}{(x-1)} + \dfrac{1}{(x-2)} - \dfrac{1}{(x-3)} - \dfrac{1}{(x-4)} - \dfrac{1}{(x-5)}\right\}.$

20. $\log y = \log \cos x + \log \cos 2x + \log \cos 3x$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} - \dfrac{2\sin 2x}{\cos 2x} - \dfrac{3\sin 3x}{\cos 3x}$

$\Rightarrow \dfrac{dy}{dx} = y \cdot \{-\tan x - 2\tan 2x - 3\tan 3x\}.$

21. $\log y = 5\log x + \dfrac{1}{2}\log(x+4) - 2\log(2x+3)$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \left\{\dfrac{5}{x} + \dfrac{1}{2(x+4)} - \dfrac{4}{(2x+3)}\right\}.$

22. $\log y = 2\log(x+1) + \dfrac{1}{2}\log(x-1) - 3\log(x+4) - x$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \left\{\dfrac{2}{(x+1)} + \dfrac{1}{2(x-1)} - \dfrac{3}{(x+4)} - 1\right\}.$

25. $\log y = \dfrac{1}{2} \cdot \{\log(x-2) + \log(2x-3) + \log(3x-4)\}$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{1}{2}\left\{\dfrac{1}{(x-2)} + \dfrac{2}{(2x-3)} + \dfrac{3}{(3x-4)}\right\}.$

26. $\log y = \log \sin 2x + \log \sin 3x + \log \sin 4x$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \left\{\dfrac{2\cos 2x}{\sin 2x} + \dfrac{3\cos 3x}{\sin 3x} + \dfrac{4\cos 4x}{\sin 4x}\right\}.$

29. $\log y = \log x + \log \cos^{-1} x - \dfrac{1}{2}\log(1-x^2)$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \left\{\dfrac{1}{x} - \dfrac{1}{\cos^{-1} x} \cdot \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2} \times \dfrac{(-2x)}{(1-x^2)}\right\}$

$\Rightarrow \dfrac{dy}{dx} = y \cdot \left\{\dfrac{1}{x} - \dfrac{1}{(\cos^{-1} x)(\sqrt{1-x^2})} + \dfrac{x}{(1-x^2)}\right\}.$

30. $\log y = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^6)$

$\Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \left\{\dfrac{1}{(1+x)} + \dfrac{2x}{(1+x^2)} + \dfrac{4x^3}{(1+x^4)} + \dfrac{6x^5}{(1+x^6)}\right\}.$

31. Let $y = u - v$, where $u = x^x$ and $v = 2^{\sin x}$.

Then, $\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}.$

Now, $u = x^x \Rightarrow \log u = x \log x$

$\Rightarrow \dfrac{1}{u} \cdot \dfrac{du}{dx} = \left(x \cdot \dfrac{1}{x} + \log x \cdot 1\right)$

$\Rightarrow \dfrac{du}{dx} = u(1 + \log x) = x^x(1 + \log x).$

And, $v = 2^{\sin x} \Rightarrow \log v = (\sin x)(\log 2)$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = (\log 2)(\cos x)$$

$$\Rightarrow \frac{dv}{dx} = v(\log 2)(\cos x) = 2^{\sin x}(\log 2)(\cos x).$$

32. Let $y = u + v$, where $u = (\log x)^x$ and $v = x^{\log x}$.

Then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Now, $u = (\log x)^x \Rightarrow \log u = x \log(\log x)$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\cdot\left\{\frac{1}{\log x} + \log(\log x)\right\}$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\cdot\left\{\frac{1}{\log x} + \log(\log x)\right\}.$$

And, $v = x^{\log x} \Rightarrow \log v = (\log x)(\log x) = (\log x)^2$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{2(\log x)}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\frac{2(\log x)}{x} = e^{(\log x)^2}\cdot\frac{2(\log x)}{x}.$$

34. Let $y = u + v$, where $u = (x\cos x)^x$ and $v = (x\sin x)^{1/x}$.

Then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Now, $u = (x\cos x)^x \Rightarrow \log u = x\log(x\cos x)$

$$\Rightarrow \log u = x\cdot\{\log x + \log\cos x\}$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\left\{\frac{1}{x} - \frac{\sin x}{\cos x}\right\} + \{\log x + \log\cos x\}\cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\{(1 - x\tan x) + (\log x + \log\cos x)\}$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\{(1 - x\tan x + \log x + \log\cos x)\}.$$

And, $v = (x\sin x)^{1/x} \Rightarrow \log v = \frac{1}{x}\log(x\sin x)$

$$\Rightarrow \log v = \frac{1}{x}\cdot\{\log x + \log\sin x\}$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\left(\frac{1}{x} + \frac{\cos x}{\sin x}\right) + (\log x + \log\sin x)\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \left\{\frac{1}{x^2} + \frac{1}{x}\cot x - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}$$

$$\Rightarrow \frac{dv}{dx} = v\cdot\left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{1/x}\cdot\left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log\sin x}{x^2}\right\}.$$

35. Let $y = u + v$, where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$.

Now, $u = (\sin x)^x \Rightarrow \log u = x\log\sin x$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{\cos x}{\sin x} + \log\sin x\cdot 1$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \cdot \{x \cot x + \log \sin x\}.$$

And, $v = \sin^{-1}\sqrt{x} \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x - x^2}}$.

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

36. Let $u = x^{x \cos x}$ and $v = \left(\frac{x^2+1}{x^2-1}\right)$. Then,

$\log u = (x \cos x) \log x$

$\Rightarrow \frac{1}{u} \cdot \frac{du}{dx} = (x \cos x) \cdot \frac{1}{x} + (\log x)(-x \sin x + \cos x)$

$\Rightarrow \frac{du}{dx} = u \cdot \{\cos x - x \sin x (\log x) + \cos x (\log x)\}$

$\Rightarrow \frac{du}{dx} = x^{x \cos x} \cdot \{\cos x - x \sin x (\log x) + \cos x (\log x)\}$.

And, $\frac{dv}{dx} = \frac{(x^2-1)\cdot 2x - (x^2+1)\cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$.

37. $\log y = x + 3 \log \sin x + 4 \log \cos x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left\{1 + \frac{3\cos x}{\sin x} - \frac{4}{\cos x} \cdot \sin x\right\}$.

38. $\log y = x \log 2 + 3x + \log \sin 4x$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = (\log 2) + 3 + \frac{4 \cos 4x}{\sin 4x}$.

39. $\log y = x \log x + (2x + 5)$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left(x \cdot \frac{1}{x} + \log x\right) + 2 = (3 + \log x)$.

40. $\log y = 5 \log (2x + 3) + 7 \log (3x - 5) + 3 \log (5x - 1)$

$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left\{\frac{5}{(2x+3)} \cdot 2 + \frac{7}{(3x-5)} \cdot 3 + \frac{3}{(5x-1)} \cdot 5\right\}$.

41. $(\cos x)^y = (\cos y)^x$

$\Rightarrow y \log \cos x = x \log \cos y$

$\Rightarrow y \cdot \left(\frac{-\sin x}{\cos x}\right) + (\log \cos x)\frac{dy}{dx} = x \cdot \left(\frac{-\sin y}{\cos y}\right) \cdot \frac{dy}{dx} + (\log \cos y) \cdot 1$

$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = (\log \cos y + y \tan x)$.

44. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \Rightarrow y(\sqrt{1-x^2}) = \sin^{-1} x$

$\Rightarrow y \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) + \sqrt{1-x^2} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{-xy}{\sqrt{1-x^2}} + \sqrt{1-x^2} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow -xy + (1-x^2)\frac{dy}{dx} = 1 \Rightarrow (1-x^2)\frac{dy}{dx} = (xy + 1)$.

45. $y^2 = (x + y) \Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}$

$\Rightarrow (2y - 1)\dfrac{dy}{dx} = 1.$

46. $x^a y^b = (x + y)^{(a+b)} \Rightarrow a\log x + b\log y = (a + b)\log(x + y).$

$\therefore \dfrac{a}{x} + \dfrac{b}{y}\cdot\dfrac{dy}{dx} = (a + b)\cdot\dfrac{1}{(x + y)}\left(1 + \dfrac{dy}{dx}\right)$

$\Rightarrow \left(\dfrac{a + b}{x + y} - \dfrac{b}{y}\right)\dfrac{dy}{dx} = \left\{\dfrac{a}{x} - \dfrac{a + b}{x + y}\right\}.$

60. Let $u = x^{(x^2-3)}$ and $v = (x - 3)^{x^2}$. Then, $y = u + v.$

Now, $u = x^{(x^2-3)}$

$\Rightarrow \log u = (x^2 - 3)\log x$

$\Rightarrow \dfrac{1}{u}\cdot\dfrac{du}{dx} = (x^2 - 3)\cdot\dfrac{1}{x} + (2x)\log x$

$\Rightarrow \dfrac{du}{dx} = u\cdot\left[\dfrac{(x^2 - 3)}{x} + (2x)\log x\right] = x^{(x^2-3)}\cdot\left\{\dfrac{(x^2 - 3)}{x} + (2x)\log x\right\}.$

And, $v = (x - 3)^{x^2}$

$\Rightarrow \log v = x^2\log(x - 3)$

$\Rightarrow \dfrac{1}{v}\cdot\dfrac{dv}{dx} = x^2\cdot\dfrac{1}{(x - 3)} + 2x\log(x - 3)$

$\Rightarrow \dfrac{dv}{dx} = v\cdot\left\{\dfrac{x^2}{(x - 3)} + 2x\log(x - 3)\right\} = (x - 3)^{x^2}\left\{\dfrac{x^2}{(x - 3)} + 2x\log(x - 3)\right\}.$

61. $y = \left(\dfrac{3 + x}{1 + x}\right)^{2\,|\,3x} \Rightarrow \log y = (2 + 3x)[\log(3 + x) - \log(1 + x)]$

$\Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = (2 + 3x)\left[\dfrac{1}{3 + x} - \dfrac{1}{1 + x}\right] + [\log(3 + x) - \log(1 + x)]\times 3.$

$\therefore f'(x) = \dfrac{dy}{dx} = \left(\dfrac{3 + x}{1 + x}\right)^{2\,|\,3x}\cdot\left\{\dfrac{(2 + 3x)(-2)}{(3 + x)(1 + x)} + 3\log\left(\dfrac{3 + x}{1 + x}\right)\right\}.$

Hence, $f'(0) = 3^2\cdot\left(\dfrac{-4}{3} + 3\log 3\right) = -12 + 27\log 3.$

7. Derivatives of an Infinite Series

If we take out a single term from an infinite series, it remains unaffected. We utilize this result in finding the derivative of an infinite series.

SOLVED EXAMPLES

EXAMPLE 1 *If* $y = x^{x^{x\ldots\infty}}$, *prove that* $\dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y\log x)}.$

SOLUTION We know that an infinite series is not affected by the exclusion of a single term.

So, we may write the given function as $y = x^y$.

Now, $y = x^y \Rightarrow \log y = y \log x$. … (i)

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{(1 - y \log x)}{y} \cdot \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \left\{\frac{y}{x} \times \frac{y}{(1 - y \log x)}\right\} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}.$$

EXAMPLE 2 *If* $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \text{ to } \infty}}}$, *prove that* $\frac{dy}{dx} = \frac{\cos x}{(2y - 1)}$.

SOLUTION We may write the given series as

$$y = \sqrt{\sin x + y} \Rightarrow y^2 = (\sin x + y). \quad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y - 1) \cdot \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(2y - 1)}.$$

EXAMPLE 3 *If* $y = e^{x + e^{x + e^{x + \ldots \text{ to } \infty}}}$, *prove that* $\frac{dy}{dx} = \frac{y}{(1 - y)}$.

SOLUTION We may write the given series as

$$y = e^{x + y} \Rightarrow \log y = (x + y). \quad \ldots \text{(i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{(1 - y)}{y} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{(1 - y)}.$$

EXAMPLE 4 *If* $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x}) \ldots \infty}}$, *prove that* $x\left(\frac{dy}{dx}\right) = \frac{y^2}{(2 - y \log x)}$.

SOLUTION We may write the given series as

$$y = (\sqrt{x})^y \Rightarrow y = x^{y/2}$$

$$\Rightarrow \log y = \frac{y}{2} \cdot \log x. \qquad \text{... (i)}$$

On differentiating both sides of (i) w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{y}{2} \cdot \frac{1}{x} + \frac{1}{2} \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \frac{1}{2}\log x\right) \cdot \frac{dy}{dx} = \frac{y}{2x}$$

$$\Rightarrow \frac{(2 - y \log x)}{2y} \cdot \frac{dy}{dx} = \frac{y}{2x}$$

$$\Rightarrow x \cdot \frac{dy}{dx} = \frac{y^2}{(2 - y \log x)}.$$

EXERCISE 10G

1. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$, prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{(1 - y \log \sin x)}$.

2. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, prove that $\frac{dy}{dx} = \frac{-y^2 \tan x}{(1 - y \log \cos x)}$.

3. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{(2y - 1)}$.

4. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{(1 - 2y)}$.

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{(2y - 1)}$.

6. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$, show that $(2y - 1) \cdot \frac{dy}{dx} = \frac{1}{x}$.

7. If $y = a^{x\,a^{x\ldots\infty}}$, prove that $\frac{dy}{dx} = \frac{y^2(\log y)}{x\,[1 - y(\log x)\,(\log y)]}$.

8. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{y}{(2y - x)}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10G)

1. $y = (\sin x)^y$. 3. $y = \sqrt{x + y} \Rightarrow y^2 = x + y$.

7. $y = a^{(x^y)} \Rightarrow \log y = x^y (\log a) \Rightarrow \log(\log y) = (y \log x) + \log(\log a)$.

8. $y = x + \frac{1}{y}$.

8. Derivative of One Function With Respect To Another Function

Let $f(x)$ and $g(x)$ be two functions of x. In order to find the derivative of $f(x)$ with respect to $g(x)$, we put $u = f(x)$ and $v = g(x)$. Now, find $\frac{du}{dv} = \frac{(du/dx)}{(dv/dx)}$, which is the required derivative.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate* e^x *w.r.t.* $\sqrt{x}$.

SOLUTION Let $u = e^x$ and $v = \sqrt{x}$.

Then, $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{e^x}{(1/2\sqrt{x})} = 2e^x\sqrt{x}.$

EXAMPLE 2 *Differentiate* $\sin^2 x$ *w.r.t.* $e^{\cos x}$.

SOLUTION Let $u = \sin^2 x$ and $v = e^{\cos x}$. Then,

$$\frac{du}{dx} = 2\sin x \cos x$$

and $\frac{dv}{dx} = e^{\cos x} \cdot (-\sin x) = (-\sin x) \cdot e^{\cos x}$.

$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{2\sin x \cos x}{(-\sin x)e^{\cos x}} = \frac{-2\cos x}{e^{\cos x}}$.

EXAMPLE 3 *Differentiate* $\sin^{-1}x$ *w.r.t.* $\tan^{-1}x$.

SOLUTION Let $u = \sin^{-1}x$ and $v = \tan^{-1}x$.

Then, $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $\frac{dv}{dx} = \frac{1}{(1+x^2)}$.

$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{1}{\sqrt{1-x^2}} \times (1+x^2) = \frac{(1+x^2)}{\sqrt{1-x^2}}$.

EXAMPLE 4 *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w.r.t.* $\tan^{-1}x$.

SOLUTION Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$

Now, $v = \tan^{-1}x \Rightarrow x = \tan v$.

Putting $x = \tan v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2 v}-1}{\tan v}\right\} = \tan^{-1}\left(\frac{\sec v - 1}{\tan v}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos v}{\sin v}\right) = \tan^{-1}\left\{\frac{2\sin^2(v/2)}{2\sin(v/2)\cos(v/2)}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{v}{2}\right\} = \frac{v}{2}.$$

$$\therefore \quad u = \frac{v}{2} \;\Rightarrow\; \frac{du}{dv} = \frac{1}{2}.$$

EXAMPLE 5 *Differentiate* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *w.r.t.* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

SOLUTION Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta,$$

$$v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta.$$

$$\therefore \quad u = v \;\Rightarrow\; \frac{du}{dv} = 1.$$

EXAMPLE 6 *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *w.r.t.* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. **[CBSE 2004C]**

SOLUTION Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Putting $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta,$$

$$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

$$\therefore \quad u = v \;\Rightarrow\; \frac{du}{dv} = 1.$$

EXAMPLE 7 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ *w.r.t.* $\cos^{-1}x^2$.

SOLUTION Let $u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1}x^2$.

Then, $\cos^{-1}x^2 = v \;\Rightarrow\; x^2 = \cos v$.

Putting $x^2 = \cos v$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos v}-\sqrt{1-\cos v}}{\sqrt{1+\cos v}+\sqrt{1-\cos v}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2(v/2)}-\sqrt{2\sin^2(v/2)}}{\sqrt{2\cos^2(v/2)}+\sqrt{2\sin^2(v/2)}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos(v/2) - \sin(v/2)}{\cos(v/2) + \sin(v/2)}\right\} = \tan^{-1}\left\{\frac{1 - \tan(v/2)}{1 + \tan(v/2)}\right\}$$

[dividing num. and denom. by $\cos(v/2)$]

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{v}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{v}{2}\right).$$

$$\therefore \quad u = \left(\frac{\pi}{4} - \frac{v}{2}\right) \Rightarrow \frac{du}{dv} = \frac{-1}{2}.$$

EXERCISE 10H

1. Differentiate x^6 with respect to $(1/\sqrt{x})$.
2. Differentiate $\log x$ with respect to $\cot x$.
3. Differentiate $e^{\sin x}$ with respect to $\cos x$.
4. Differentiate $\tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1}x^2$.
5. Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.
6. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}(2x^2-1)$.
7. Differentiate $\sin^3 x$ with respect to $\cos^3 x$.
8. Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.
9. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.
10. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **[CBSE 2014]**

ANSWERS (EXERCISE 10H)

1. $-12x^{13/2}$ **2.** $\frac{-\sin^{-1}x}{x}$ **3.** $-e^{\sin x}\cot x$ **4.** $\frac{1}{2}$ **5.** 1

6. $-\frac{1}{2}$ **7.** $-\tan x$ **8.** $\frac{2}{3}$ **9.** $\frac{1}{4}$ **10.** $-\frac{1}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10H)

4. Put $x^2 = \cos v$. 8. Put $x = \tan\theta$. 10. Put $x = \cos\theta$.

9. Derivatives of Parametric Functions

Sometimes x and y are given as functions of a variable t. Then, t is called a *parameter*.

Let $x = f(t)$ and $y = g(t)$. Then,

$\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$.

$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}$, where $f'(t) \neq 0$.

SUMMARY

Let $x = f(t)$ and $y = g(t)$. Then,

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{g'(t)}{f'(t)}, \text{ where } f'(t) \neq 0.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find* $\frac{dy}{dx}$, *when* $x = a(t + \sin t)$ *and* $y = a(1 - \cos t)$.

SOLUTION We have:

$$x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t);$$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = \frac{a \sin t}{a(1 + \cos t)} = \frac{2a \sin (t/2) \cos (t/2)}{2a \cos^2 (t/2)} = \tan \frac{t}{2}.$$

EXAMPLE 2 *If* $x = a[\cos \theta + \log \tan (\theta/2)]$ *and* $y = a \sin \theta$, *find* $\frac{dy}{dx}$ *at* $\theta = (\pi/4)$.

[CBSE 2008]

SOLUTION We have

$$x = a\{\cos \theta + \log \tan (\theta/2)\}$$

$$\Rightarrow \frac{dx}{d\theta} = a\left\{-\sin \theta + \frac{\sec^2 (\theta/2)}{2 \tan (\theta/2)}\right\} = a\left\{-\sin \theta + \frac{1}{2 \sin (\theta/2) \cos (\theta/2)}\right\}$$

$$= a\left\{-\sin \theta + \frac{1}{\sin \theta}\right\} = \frac{a(1 - \sin^2 \theta)}{\sin \theta} = \frac{a \cos^2 \theta}{\sin \theta}.$$

And, $y = a \sin \theta \Rightarrow \frac{dy}{d\theta} = a \cos \theta$.

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{d\theta} \times \frac{d\theta}{dx}\right) = \left(a \cos \theta \cdot \frac{\sin \theta}{a \cos^2 \theta}\right) = \tan \theta.$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\theta = \pi/4} = \tan \frac{\pi}{4} = 1.$$

EXAMPLE 3 *If* $x = a\sin 2t(1+\cos 2t)$ *and* $y = b\cos 2t(1-\cos 2t)$, *show that* $\left(\frac{dy}{dx}\right)_{at\ t=\frac{\pi}{4}} = \frac{b}{a}$. **[CBSE 2006]**

SOLUTION We have

$$x = a\sin 2t(1+\cos 2t)$$

$$\Rightarrow \frac{dx}{dt} = a\cdot[\sin 2t(-2\sin 2t) + (1+\cos 2t)(2\cos 2t)]$$

$$= (2a)\cdot[-\sin^2 2t + \cos 2t + \cos^2 2t]$$

$$= (2a)[\cos 4t + \cos 2t] \qquad \{\because\ (\cos^2 2t - \sin^2 2t) = \cos 4t\}.$$

And, $y = b\cos 2t(1-\cos 2t)$

$$\Rightarrow \frac{dy}{dt} = b[\cos 2t(2\sin 2t) + (1-\cos 2t)(-2\sin 2t)]$$

$$= (2b)[\sin 2t\cos 2t - \sin 2t + \sin 2t\cos 2t]$$

$$= (2b)[2\sin 2t\cos 2t - \sin 2t] = (2b)[\sin 4t - \sin 2t].$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{(2b)(\sin 4t - \sin 2t)}{(2a)(\cos 4t + \cos 2t)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(t=\frac{\pi}{4}\right)} = \frac{b}{a}\cdot\frac{\left\{\sin\left(4\times\frac{\pi}{4}\right) - \sin\left(2\times\frac{\pi}{4}\right)\right\}}{\left\{\cos\left(4\times\frac{\pi}{4}\right) + \cos\left(2\times\frac{\pi}{4}\right)\right\}}$$

$$= \frac{b}{a}\cdot\frac{\left(\sin\pi - \sin\frac{\pi}{2}\right)}{\left(\cos\pi + \cos\frac{\pi}{2}\right)} = \frac{b}{a}\cdot\frac{(0-1)}{(-1+0)} = \frac{b}{a}.$$

EXAMPLE 4 *If* $x = a\left(\frac{1+t^2}{1-t^2}\right)$ *and* $y = \frac{2t}{(1-t^2)}$, *find* $\frac{dy}{dx}$. **[CBSE 2005]**

SOLUTION We have

$$x = a\left[-1 + \frac{2}{(1-t^2)}\right] = a[-1 + 2(1-t^2)^{-1}]$$

$$\Rightarrow \frac{dx}{dt} = a[0 + 2(-1)(1-t^2)^{-2}(-2t)] = a\times\frac{4t}{(1-t^2)^2} = \frac{4at}{(1-t^2)^2}.$$

And, $y = \frac{2t}{(1-t^2)}$

$$\Rightarrow \frac{dy}{dt} = \frac{(1-t^2)\cdot 2 - 2t(-2t)}{(1-t^2)^2} = \frac{2(1+t^2)}{(1-t^2)^2}.$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \left\{\frac{2(1+t^2)}{(1-t^2)^2}\times\frac{(1-t^2)^2}{4at}\right\} = \frac{(1+t^2)}{2at}.$$

EXAMPLE 5 *If* $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, *find* $\dfrac{d^2y}{dx^2}$ *at* $t = \dfrac{\pi}{3}$.

[CBSE 2005C]

SOLUTION We have

$$x = 3\sin t - \sin 3t$$

$$\Rightarrow \frac{dx}{dt} = 3\cos t - 3\cos 3t. \quad \text{... (i)}$$

And, $y = 3\cos t - \cos 3t$

$$\Rightarrow \frac{dy}{dt} = -3\sin t + 3\sin 3t. \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t} = \frac{\sin 3t - \sin t}{\cos t - \cos 3t}$$

$$= \frac{2\cos 2t \sin t}{2\sin 2t \sin t} = \cot 2t.$$

$$\therefore \frac{d^2y}{dx^2} = -2\text{cosec}^2 2t \cdot \frac{dt}{dx} = \frac{-2\text{cosec}^2 2t}{(dx/dt)} = \frac{-2\text{cosec}^2 2t}{3(\cos t - \cos 3t)}.$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\left(t=\frac{\pi}{3}\right)} = \frac{-2\text{cosec}^2(2\pi/3)}{3\cos\frac{\pi}{3} - \cos\pi} = -2 \times \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{3\left(\frac{1}{2}+1\right)}$$

$$= \left(-2 \times \frac{4}{3} \times \frac{2}{9}\right) = \frac{-16}{27}.$$

EXAMPLE 6 *If* $x = \sqrt{a^{\sin^{-1}t}}$ *and* $y = \sqrt{a^{\cos^{-1}t}}$, *show that* $\dfrac{dy}{dx} = \dfrac{-y}{x}$.

SOLUTION We have

$$x^2 = a^{\sin^{-1}t} \text{ and } y^2 = a^{\cos^{-1}t}$$

$$\Rightarrow 2x\frac{dx}{dt} = a^{\sin^{-1}t} \cdot \frac{1}{\sqrt{1-t^2}} \text{ and } 2y\frac{dy}{dt} = a^{\cos^{-1}t} \cdot \frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{2y}{2x} \cdot \frac{(dy/dt)}{(dx/dt)} = \frac{-a^{\cos^{-1}t}}{\sqrt{1-t^2}} \times \frac{\sqrt{1-t^2}}{a^{\sin^{-1}t}}$$

$$\Rightarrow \frac{y}{x} \cdot \frac{dy}{dx} = -\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} = -\frac{y^2}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \qquad \text{[on dividing both sides by } \frac{y}{x}\text{]}.$$

Hence, $\dfrac{dy}{dx} = \dfrac{-y}{x}$.

EXAMPLE 7 *If* $a > 0$, $x = \left(t + \dfrac{1}{t}\right)^a$ *and* $y = a^{\left(t+\frac{1}{t}\right)}$, *find* $\dfrac{dy}{dx}$.

SOLUTION We have

$$x = \left(t + \frac{1}{t}\right)^a \Rightarrow \frac{dx}{dt} = a\left(t + \frac{1}{t}\right)^{(a-1)} \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right) = a\left(t + \frac{1}{t}\right)^{(a-1)} \cdot \left(1 - \frac{1}{t^2}\right).$$

$$\text{And, } y = a^{\left(t+\frac{1}{t}\right)} \Rightarrow \frac{dy}{dt} = a^{\left(t+\frac{1}{t}\right)} \log a \cdot \frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$= a^{\left(t+\frac{1}{t}\right)} \log a \cdot \left(1-\frac{1}{t^2}\right).$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a^{\left(t+\frac{1}{t}\right)} \log a \cdot \left(1-\frac{1}{t^2}\right)}{a\left(t+\frac{1}{t}\right)^{(a-1)}\left(1-\frac{1}{t^2}\right)} = \frac{a^{\left(t+\frac{1}{t}-1\right)} \cdot \log a}{\left(t+\frac{1}{t}\right)^{(a-1)}}.$$

EXERCISE 10I

Find $\frac{dy}{dx}$, when

1. $x = at^2,\ y = 2at$
2. $x = a\cos\theta,\ y = b\sin\theta$
3. $x = a\cos^2\theta,\ y = b\sin^2\theta$
4. $x = a\cos^3\theta,\ y = a\sin^3\theta$
5. $x = a(1-\cos\theta),\ y = a(\theta+\sin\theta)$
6. $x = a\log t,\ y = b\sin t$
7. $x = (\log t + \cos t),\ y = (e^t + \sin t)$
8. $x = \cos\theta + \cos 2\theta,\ y = \sin\theta + \sin 2\theta$
9. $x = \sqrt{\sin 2\theta},\ y = \sqrt{\cos 2\theta}$
10. $x = e^{\theta}(\sin\theta + \cos\theta),\ y = e^{\theta}(\sin\theta - \cos\theta)$
11. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$ **[CBSE 2003C]**
12. $x = \frac{3at}{(1+t^2)},\ y = \frac{3at^2}{(1+t^2)}$
13. $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$
14. $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}},\ y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$
15. If $x = 3\cos t - 2\cos^3 t,\ y = 3\sin t - 2\sin^3 t$, show that $\frac{dy}{dx} = \cot t$.
16. If $x = \frac{1+\log t}{t^2}$ and $y = \frac{3+2\log t}{t}$, show that $\frac{dy}{dx} = t$.
17. If $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$. **[CBSE 2003C]**
18. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, show that $\frac{dy}{dx} = \tan\frac{3\theta}{2}$. **[CBSE 2013C]**
19. If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, find $\frac{dy}{dx}$.
20. If $x = (2\cos\theta - \cos 2\theta)$ and $y = (2\sin\theta - \sin 2\theta)$, find $\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}}$. **[CBSE 2005C]**
21. If $x = a(\theta - \sin\theta),\ y = a(1+\cos\theta)$, find $\frac{d^2y}{dx^2}$. **[CBSE 20011]**

ANSWERS (EXERCISE 10I)

1. $\frac{1}{t}$ **2.** $\frac{-b\cot\theta}{a}$ **3.** $\frac{-b}{a}$ **4.** $-\tan\theta$

5. $\cot\frac{\theta}{2}$ **6.** $\frac{bt\cos t}{a}$ **7.** $\frac{t(e^t+\cos t)}{(1-t\sin t)}$ **8.** $-\left(\frac{\cos\theta+2\cos 2\theta}{\sin\theta+2\sin 2\theta}\right)$

9. $-(\tan 2\theta)^{3/2}$ **10.** $\tan\theta$ **11.** $\tan\theta$ **12.** $\frac{2t}{(1-t^2)}$

13. $\frac{(t^2-1)}{2t}$ **14.** 1 **17.** 1 **19.** $-\cot 3t$ **20.** $\frac{-3}{2}$ **21.** $\frac{1}{4a}\text{cosec}^4\frac{\theta}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 10I)

14. Put $t=\tan\theta$.

19. $\log x = 3\log\sin t - \frac{1}{2}\log\cos 2t$ and $\log y = 3\log\cos t - \frac{1}{2}\log\cos 2t$

$\Rightarrow \frac{1}{x}\cdot\frac{dx}{dt} = \frac{3}{\sin t}\cdot\cos t - \frac{1}{2}\cdot\frac{1}{\cos 2t}(-2\sin 2t), \frac{dy}{dt} = y\cdot(-3\tan t + \tan 2t)$

$\Rightarrow \frac{dx}{dt} = x\cdot[3\cot t + \tan 2t],\ \frac{dy}{dt} = y\cdot[-3\tan t + \tan 2t]$

$$\Rightarrow \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{y}{x}\cdot\frac{(-3\tan t + \tan 2t)}{(3\cot t + \tan 2t)} = \frac{\cot^3 t\cdot\left(-3\tan t + \frac{2\tan t}{1-\tan^2 t}\right)}{\left(\frac{3}{\tan t} + \frac{2\tan t}{1-\tan^2 t}\right)}$$

$$= \frac{-(1-3\tan^2 t)}{(3\tan t - \tan^3 t)} = \frac{-1}{\tan 3t} = -\cot 3t.$$

20. $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \tan\frac{3\theta}{2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{d\theta}{dx} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{1}{2(\sin 2\theta - \sin\theta)}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{3}{4}\sec^2\frac{3\pi}{4}\cdot\frac{1}{\left(\sin\pi - \sin\frac{\pi}{2}\right)} = \frac{3}{4}\times 2\times\frac{1}{(0-1)} = \frac{-3}{2}.$

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10. Second-Order Derivatives

Let $y=f(x)$ be a differentiable function of x whose second-order derivative exists. We denote the second-order derivative of y w.r.t. x by $\frac{d^2y}{dx^2}$ or y_2.

SOLVED EXAMPLES

EXAMPLE 1 *Find the second-order derivative of:*
(i) x^{10} *(ii)* $\log x$ *(iii)* $\tan^{-1} x$

SOLUTION (i) Let $y = x^{10}$. Then,

$$\frac{dy}{dx} = 10x^9.$$

$$\therefore \quad \frac{d^2y}{dx^2} = (10 \times 9)x^8 = 90x^8.$$

Hence, $\frac{d^2y}{dx^2}(x^{10}) = 90x^8.$

(ii) Let $y = \log x$. Then,

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(x^{-1}) = (-1)x^{(-1-1)} = -x^{-2} = \frac{-1}{x^2}.$$

Hence, $\frac{d^2}{dx^2}(\log x) = \frac{-1}{x^2}.$

(iii) Let $y = \tan^{-1} x$. Then,

$$\frac{dy}{dx} = \frac{1}{(1+x^2)} = (1+x^2)^{-1}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1} = (-1)(1+x^2)^{-2} \cdot (2x) = \frac{-2x}{(1+x^2)^2}.$$

Hence, $\frac{d^2}{dx^2}\{\tan^{-1} x\} = \frac{-2x}{(1+x^2)^2}.$

EXAMPLE 2 *If* $y = (\tan x + \sec x)$, *prove that* $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$

SOLUTION Given that $y = (\tan x + \sec x)$.

$$\therefore \quad \frac{dy}{dx} = \sec^2 x + \sec x \tan x = \left(\frac{1}{\cos^2 x} + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\right)$$

$$= \left(\frac{1+\sin x}{\cos^2 x}\right) = \left(\frac{1+\sin x}{1-\sin^2 x}\right) = \frac{1}{(1-\sin x)}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{(1-\sin x)}\right\} = \frac{d}{dx}(1-\sin x)^{-1}$$

$$= (-1)(1-\sin x)^{-2}(-\cos x) = \frac{\cos x}{(1-\sin x)^2}.$$

Hence, $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$

EXAMPLE 3 *If* $y = e^{4x} \sin 3x$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION Let $y = e^{4x} \sin 3x$. Then,

$$\frac{dy}{dx} = 3e^{4x}\cos 3x + 4e^{4x}\sin 3x = e^{4x}(3\cos 3x + 4\sin 3x).$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\{e^{4x}(3\cos 3x + 4\sin 3x)\}$$

$$= e^{4x}(-9\sin 3x + 12\cos 3x) + 4e^{4x}(3\cos 3x + 4\sin 3x)$$

$$= e^{4x}(7\sin 3x + 24\cos 3x).$$

EXAMPLE 4 *If* $y = (x^4 + \cot x)$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION We have

$$y = (x^4 + \cot x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 4x^3 - \text{cosec}^2 x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3 - \text{cosec}^2 x)$$

$$= 4 \cdot \frac{d}{dx}(x^3) - \frac{d}{dx}(\text{cosec}^2 x)$$

$$= (4 \times 3x^2) - 2\text{cosec}\, x(-\text{cosec}\, x \cot x)$$

$$= (12x^2 + 2\text{cosec}^2 x \cot x).$$

EXAMPLE 5 *Find the second derivative of* $\log(\log x)$ *w.r.t.* x.

SOLUTION Let $y = \log(\log x)$. Then,

$$\frac{dy}{dx} = \frac{1}{(\log x)} \cdot \frac{1}{x} = \frac{1}{(x \log x)} = (x \log x)^{-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(x \log x)^{-1}$$

$$= (-1)(x \log x)^{-2} \cdot \frac{d}{dx}(x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left(x \cdot \frac{1}{x} + \log x \cdot 1\right) = \frac{-(1 + \log x)}{(x \log x)^2}.$$

EXAMPLE 6 *If* $y = \sin(\log x)$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION We have

$$y = \sin(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\{\sin(\log x)\} = \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{\cos(\log x)}{x}\right\}$$

$$= \frac{x \cdot \frac{d}{dx}\{\cos(\log x)\} - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left\{-\sin(\log x) \cdot \frac{1}{x}\right\} - \cos(\log x) \cdot 1}{x^2}$$

$$= \frac{-\{\sin(\log x) + \cos(\log x)\}}{x^2}.$$

EXAMPLE 7 *If* $e^y(x+1) = 1$, *prove that* $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

SOLUTION We have

$$e^y(x+1) = 1 \Rightarrow e^y = \frac{1}{(x+1)} \quad \text{... (i)}$$

$$\Rightarrow y = \log\left\{\frac{1}{(x+1)}\right\} = \log 1 - \log(x+1)$$

$$\Rightarrow y = -\log(x+1). \quad \text{... (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{(x+1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{dy}{dx}\right)^2.$$

Hence, $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

EXAMPLE 8 *If* $y = A\cos nx + B\sin nx$, *prove that* $\frac{d^2y}{dx^2} + n^2y = 0$.

SOLUTION We have

$$y = A\cos nx + B\sin nx$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(A\cos nx) + \frac{d}{dx}(B\sin nx)$$

$$= -An\sin nx + Bn\cos nx$$

$$= n(B\cos nx - A\sin nx)$$

$$\Rightarrow \frac{d^2y}{dx^2} = n \cdot \frac{d}{dx}(B\cos nx - A\sin nx)$$

$$= n \cdot \left\{B \cdot \frac{d}{dx}(\cos nx) - A \cdot \frac{d}{dx}(\sin nx)\right\}$$

$$= n \cdot \{-Bn\sin nx - An\cos nx\}$$

$$= -n^2(A\cos nx + B\sin nx) = -n^2y$$

$$\Rightarrow \frac{d^2y}{dx^2} + n^2y = 0.$$

EXAMPLE 9 *If* $y = e^x(\sin x + \cos x)$, *prove that* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. **[CBSE 2002, '09]**

SOLUTION We have

$$y = e^x(\sin x + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x(\cos x - \sin x) + (\sin x + \cos x) \cdot e^x = 2e^x \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \cdot \frac{d}{dx}(e^x \cos x)$$

$$= 2 \cdot \left\{ e^x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(e^x) \right\}$$

$$= 2 \cdot \{e^x(-\sin x) + (\cos x)e^x\} = 2e^x(\cos x - \sin x).$$

$$\therefore \left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y \right)$$

$$= 2e^x(\cos x - \sin x) - 4e^x \cos x + 2e^x(\sin x + \cos x) = 0.$$

Hence, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$

EXAMPLE 10 *If* $y = 3e^{2x} + 2e^{3x}$, *prove that* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

SOLUTION We have

$$y = 3e^{2x} + 2e^{3x} \quad \text{... (i)}$$

$$\Rightarrow \frac{dy}{dx} = 3 \cdot \frac{d}{dx}(e^{2x}) + 2 \cdot \frac{d}{dx}(e^{3x}) \text{ [on differentiating (i) w.r.t. } x\text{]}$$

$$= (3 \times 2e^{2x}) + (2 \times 3e^{3x}) = (6e^{2x} + 6e^{3x})$$

$$\Rightarrow \frac{dy}{dx} = 6(e^{2x} + e^{3x}). \quad \text{... (ii)}$$

On differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6 \cdot \left\{ \frac{d}{dx}(e^{2x}) + \frac{d}{dx}(e^{3x}) \right\}$$

$$= 6 \cdot (2e^{2x} + 3e^{3x}).$$

$$\therefore \left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y \right)$$

$$= 6(2e^{2x} + 3e^{3x}) - 30(e^{2x} + e^{3x}) + (18e^{2x} + 12e^{3x})$$

$$= (12 - 30 + 18)e^{2x} + (18 - 30 + 12)e^{3x} = 0.$$

Hence, $\left(\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y \right) = 0.$

EXAMPLE 11 *If* $y = \sin^{-1} x$, *prove that* $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$. **[CBSE 2012]**

SOLUTION Given: $y = \sin^{-1}x$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1^2 = \frac{1}{(1-x^2)}$$

$$\Rightarrow \quad (1-x^2)y_1^2 = 1. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$(1-x^2)\cdot 2y_1y_2 + y_1^2(-2x) = 0$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = 0.$$

Hence, $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0.$

EXAMPLE 12 *If* $y = (\tan^{-1}x)^2$, *prove that* $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2$. **[CBSE 2012]**

SOLUTION Given: $y = (\tan^{-1}x)^2$. ... (i)

On differentiating both sides of (i) w.r.t. x, we get

$$y_1 = 2\tan^{-1}x \cdot \frac{1}{(1+x^2)}$$

$$\Rightarrow \quad (1+x^2)y_1 = 2\tan^{-1}x$$

$$\Rightarrow \quad (1+x^2)^2y_1^2 = 4(\tan^{-1}x)^2 \qquad \text{[on squaring both sides]}$$

$$\Rightarrow \quad (1+x^2)^2y_1^2 - 4y = 0. \qquad \text{... (ii)}$$

On differentiating both sides of (ii) w.r.t. x, we get

$$(1+x^2)^2\cdot 2y_1y_2 + y_1^2\cdot 2(1+x^2)\cdot 2x - 4y_1 = 0$$

$$\Rightarrow \quad (1+x^2)^2y_2 + 2x(1+x^2)y_1 - 2 = 0.$$

Hence, $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2.$

EXAMPLE 13 *If* $x = a(\theta + \sin\theta)$ *and* $y = a(1-\cos\theta)$, *find* $\dfrac{d^2y}{dx^2}$ *at* $\theta = \dfrac{\pi}{2}$.

[CBSE 2005, '09]

SOLUTION We have

$$x = a(\theta + \sin\theta) \text{ and } y = a(1-\cos\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a(1+\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)}$$

$$= \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan\frac{\theta}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{d\theta}{dx} = \left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)\times\frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{2}\sec^2\frac{\pi}{4}\cdot\frac{1}{a\left(1+\cos\frac{\pi}{2}\right)} = \frac{1}{a}.$$

EXAMPLE 14 *If* $x = (2\cos\theta - \cos 2\theta)$ *and* $y = (2\sin\theta - \sin 2\theta)$, *find* $\frac{d^2y}{dx^2}$ *at* $\theta = \frac{\pi}{2}$.

[CBSE 2005C]

SOLUTION We have

$$x = (2\cos\theta - \cos 2\theta) \text{ and } y = (2\sin\theta - \sin 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = (-2\sin\theta + 2\sin 2\theta) \text{ and } \frac{dy}{d\theta} = (2\cos\theta - 2\cos 2\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{(2\cos\theta - 2\cos 2\theta)}{(-2\sin\theta + 2\sin 2\theta)} = \frac{(\cos\theta - \cos 2\theta)}{(\sin 2\theta - \sin\theta)}$$

$$= \frac{2\sin\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}}{2\cos\left(\frac{3\theta}{2}\right)\sin\frac{\theta}{2}} = \tan\frac{3\theta}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{3\theta}{2}\right) = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{d\theta}{dx} = \frac{3}{2}\sec^2\frac{3\theta}{2}\cdot\frac{1}{2(\sin 2\theta - \sin\theta)}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{3}{2}\cdot\sec^2\left(\frac{3\pi}{4}\right)\cdot\frac{1}{\left(2\sin\pi - \sin\frac{\pi}{2}\right)}$$

$$= \frac{-3}{4}\sec^2\frac{\pi}{4} = \frac{-3}{4}\times(\sqrt{2})^2 = \frac{-3}{2}$$

$$\left[\because \sec\frac{3\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right) = -\sec\frac{\pi}{4}\right].$$

EXERCISE 10J

1. Find the second derivative of:
   (i) x^{11} (ii) 5^x (iii) $\tan x$ (iv) $\cos^{-1}x$
2. Find the second derivative of:
   (i) $x\sin x$ (ii) $e^{2x}\cos 3x$ (iii) $x^3\log x$
3. If $y = x + \tan x$, show that $\cos^2 x\cdot\frac{d^2y}{dx^2} - 2y + 2x = 0$.
4. If $y = 2\sin x + 3\cos x$, show that $y + \frac{d^2y}{dx^2} = 0$.

5. If $y = 3\cos(\log x) + 4\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$. **[CBSE 2009, '12]**

6. If $y = e^{-x}\cos x$, show that $\frac{d^2y}{dx^2} = 2e^{-x}\sin x$.

7. If $y = \sec x - \tan x$, show that $(\cos x)\frac{d^2y}{dx^2} = y^2$.

8. If $y = (\text{cosec } x + \cot x)$, prove that $(\sin x)\frac{d^2y}{dx^2} - y^2 = 0$.

9. If $y = \tan^{-1}x$, show that $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

10. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0$.

11. If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$. **[CBSE 2009]**

12. Find the second derivative of $e^{3x}\sin 4x$.

13. Find the second derivative of $\sin 3x\cos 5x$.

HINT: $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

14. If $y = e^{\tan x}$, prove that $(\cos^2 x)\frac{d^2y}{dx^2} - (1 + \sin 2x)\cdot\frac{dy}{dx} = 0$. **[CBSE 2009C]**

15. If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{(2\log x - 3)}{x^3}$.

16. If $y = e^{ax}\cos bx$, show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.

17. If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

18. If $x = at^2$ and $y = 2at$, find $\frac{d^2y}{dx^2}$ at $t = 2$.

19. If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \pi$.

20. If $y = \sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. **[CBSE 2007]**

21. If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$. **[CBSE 2009]**

22. If $y = e^x\sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. **[CBSE 2009C]**

23. If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and $y = a\sin\theta$, show that the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$ is $\frac{4}{a}$. **[CBSE 2009, '13]**

24. If $x = \cos t + \log \tan \frac{t}{2}$, $y = \sin t$ then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[CBSE 2012C]**

25. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. **[CBSE 2014]**

26. If $y = (\cot^{-1}x)^2$, then show that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$. **[CBSE 2010C]**

27. If $y = \{x + \sqrt{x^2+1}\}^m$, then show that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$. **[CBSE 2013C]**

28. If $y = \log [x + \sqrt{x^2+a^2}]$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$. **[CBSE 2013C]**

29. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, show that $\frac{d^2y}{dx^2} = \frac{1}{a}\left(\frac{\sec^3\theta}{\theta}\right)$. **[CBSE 2011C, 12C]**

30. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that
$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$ **[CBSE 2014]**

ANSWERS (EXERCISE 10J)

1. (i) $110x^9$ (ii) $5^x(\log 5)^2$ (iii) $2\sec^2 x \tan x$ (iv) $\frac{-x}{(1-x^2)^{3/2}}$

2. (i) $-x\sin x + 2\cos x$ (ii) $-e^{2x}(5\cos 3x + 12\sin 3x)$ (iii) $(5x + 6x\log x)$

12. $e^{3x}(24\cos 4x - 7\sin 4x)$ **13.** $(2\sin 2x - 32\sin 8x)$

18. $\frac{-1}{16a}$ **19.** $\frac{-1}{4a}$ **24.** $\frac{-1}{\sqrt{2}}$, $2\sqrt{2}$